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Topologically Twisted SUSY Gauge Theory, Gauge-Bethe Correspondence and Quantum Cohomology

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ABSTRACT: We calculate partition function and correlation functions in A-twisted 2d $\mathcal{N} = (2, 2)$ theories and topologically twisted 3d $\mathcal{N} = 2$ theories containing adjoint chiral multiplet with particular choices of R -charges and the magnetic fluxes for flavor symmetries. According to Gauge-Bethe correspondence, they correspond to Heisenberg XXX and XXZ spin chain models. We identify the partition function as the inverse of the norm of the Bethe eigenstates. Correlation functions are identified as the coefficients of the expectation value of Baxter Q -operators. In addition, we consider correlation functions of 2d $\mathcal{N} = (2, 2)^*$ theory and their relation to equivariant quantum cohomology and equivariant integration of cotangent bundle of Grassmann manifolds. Also, we study the ring relations of supersymmetric Wilson loops in 3d $\mathcal{N} = 2^*$ theory and Bethe subalgebra of XXZ spin chain model.

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1 Introduction

The Gauge-Bethe correspondence states that quantum integrable models correspond to the supersymmetric gauge theories. The XXX Heisenberg spin chain model was considered as one of primary examples of Gauge-Bethe correspondence in the original papers [1, 2]. It was argued that the supersymmetric vacua of softly broken 2d $\mathcal{N} = (4, 4)$ theory by the mass of adjoint chiral multiplet, usually called as 2d $\mathcal{N} = (2, 2)^*$ theory, is naturally identified with Bethe ansatz equation for XXX spin chain model. Also, twisted superpotential was identified with Yang-Yang potential.

Recently, the partition function and correlation functions in topologically twisted generic 2d $\mathcal{N} = (2, 2)$ theories on S^2 [3] has been calculated, and also partition function in topologically twisted generic 3d $\mathcal{N} = 2$ theories on $S^1 \times S^2$ [4] (See also [5]) and 4d $\mathcal{N} = 1$ theories on $T^2 \times S^2$ [4, 6–8] by considering the rigid limit of supergravity.

In this paper, we study 2d $\mathcal{N} = (2, 2)$ theories and 3d $\mathcal{N} = 2$ theories containing adjoint chiral multiplet with two different choices of R -charges and background magnetic fluxes but with same gauge group and matter contents. We calculate the partition functions of A-twisted 2d $\mathcal{N} = (2, 2)$ theory on S^2 and the partition function of topologically twisted 3d $\mathcal{N} = 2$ theory on $S^1 \times S^2$ with all equivariant parameters associated to flavor symmetries but the one associated to rotational symmetry on S^2 turned off. We match them with the inverse of the norm of Bethe eigenstates by choosing a particular R -charges and background fluxes for flavor symmetries. The gauge invariant operators forms chiral ring and expectation values of them provide the coefficient of the expectation value of Baxter Q -operator. Thus with proper choice of coefficients the expectation value of the gauge invariant operators provide the expectation value of conserved charges of corresponding spin chain model.

We also explicitly calculate the correlation functions of A-twisted 2d $\mathcal{N} = (2, 2)^*$ theory whose target space (in nonlinear sigma model limit) is cotangent bundle of Grassmannians for several examples. On the other hand, we evaluate the equivariant integration of equivariant cohomology for cotangent bundle of Grassmannians [9] where they showed that Bethe subalgebra of XXX spin chain model is isomorphic to the equivariant quantum cohomology ring¹. We provide a way to calculate the equivariant integration, and by checking several examples we see that the result is consistent with the correlation functions. In the calculation, we also use Seiberg-like duality.

It was shown in [10] that Bethe subalgebra of XXZ spin chain model is given by certain generators and relations analogous in [9].² With Gauge-Bethe correspondence in mind, we see that Wilson loop algebra agree with Bethe subalgebra by checking several examples. Also, we consider Seiberg-like duality of 3d $\mathcal{N} = 2^*$ theory in the context of Bethe subalgebra.

In the final section, we conclude with summary of our results and discuss some future directions.

Note added: While this work is being completed, we heard that a paper [11], which has a partial overlap with ours, will appear.

2 Gauge/Bethe Correspondence and Bethe Norm

Given 2d $\mathcal{N} = (2, 2)$ gauge theories with the effective twisted superpotential $\widetilde{\mathcal{W}}_{\text{eff}}(\sigma)$, the condition for supersymmetric vacua is given by

$$\exp \left(2\pi i \frac{\partial \widetilde{\mathcal{W}}_{\text{eff}}(\sigma)}{\partial \sigma_a} \right) = 1, \quad (2.1)$$

and according to Gauge/Bethe correspondence [1, 2, 12], it is identified with the Bethe ansatz equation of a certain integrable model. Also, the twisted superpotential $\widetilde{\mathcal{W}}_{\text{eff}}(\sigma)$

¹They considered generic partial flag manifolds where Grassmannian is part of them.

²The Bethe subalgebra of XXZ spin chain model was conjectured to be identical to the equivariant quantum K -theory ring [10].

of 2d $\mathcal{N} = (2, 2)$ theories corresponds to the Yang-Yang potential of the corresponding integrable model. For isotropic $SU(2)$ Heisenberg spin chain model $XXX_{1/2}$, similarly for anisotropic $XXZ_{1/2}$ spin chain model, with spin-1/2 degree of freedom at each sites, the twisted mass parameters for flavor symmetries are related to the parameters for the position of the lattice sites with respect to symmetric round lattice configuration.

In this section, we relate the norm of the Bethe eigenstates of $XXX_{1/2}$ and $XXZ_{1/2}$ spin chain model to the partition function of topologically twisted 2d $\mathcal{N} = (2, 2)$ theory and 3d $\mathcal{N} = 2$ theory, respectively. We also discuss the relation between correlation functions and the coefficients of the expectation value of Baxter Q -operator as well as conserved charges.

2.1 The norm of Bethe eigenstates in $XXX_{1/2}$ and $XXZ_{1/2}$ spin chain model

We are interested in inhomogeneous $XXX_{1/2}$ and $XXZ_{1/2}$ spin chain model with M lattice sites. The monodromy matrix, $T(\lambda)$, of $XXX_{1/2}$ and $XXZ_{1/2}$ model takes a form of 2×2 matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \quad (2.2)$$

acting on the 2-dimensional auxiliary space where λ is a spectral parameter. Therefore, the transfer matrix τ obtained by the trace of monodromy matrix,

$$\tau(\lambda) = \text{Tr } T(\lambda), \quad (2.3)$$

is $\tau(\lambda) = A(\lambda) + D(\lambda)$. With quasi-periodic boundary condition $\vec{S}_{M+1} = e^{\frac{i}{2}\vartheta\sigma_3}\vec{S}_1e^{-\frac{i}{2}\vartheta\sigma_3}$ where $\vec{S}_a = \frac{1}{2}\vec{\sigma}_{(a)}$ is the generators at a -th site, the transfer matrix is given by $A(\lambda) + e^{i\vartheta}D(\lambda)$ [1, 13].

The R-matrix of $XXX_{1/2}$ and $XXZ_{1/2}$ model is

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix} \quad (2.4)$$

where

$$f(\mu, \lambda) = 1 + \frac{ic}{\mu - \lambda}, \quad g(\mu, \lambda) = \frac{ic}{\mu - \lambda} \quad (2.5)$$

for $XXX_{1/2}$ model where c is an auxiliary parameter, and

$$f(\mu, \lambda) = \frac{\sinh(\mu - \lambda + 2i\eta)}{\sinh(\mu - \lambda)}, \quad g(\mu, \lambda) = \frac{i \sin 2\eta}{\sinh(\mu - \lambda)} \quad (2.6)$$

for $XXZ_{1/2}$ model where η is related to the anisotropy parameter.

The R-matrix satisfies Yang-Baxter equation

$$(I \otimes R(\lambda, \mu)) (R(\lambda, \nu) \otimes I) (I \otimes R(\mu, \nu)) = (R(\mu, \nu) \otimes I) (I \otimes R(\lambda, \nu)) (R(\lambda, \mu) \otimes I) \quad (2.7)$$

where \otimes denotes tensor product of two matrices.³ The Yang-Baxter equation implies

$$R(\lambda, \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R(\lambda, \mu), \quad (2.8)$$

and this provides the commutation relations of the matrix elements of monodromy matrix $T(\lambda)$. Also, from (2.8) and due to trace identities, one can show that transfer matrix commutes with Hamiltonian,

$$[\tau(\lambda), H] = 0$$

so $\tau(\lambda)$ is a generating function of conserved charges. As they commute, the eigenfunction of the transfer matrix is also the eigenfunction of Hamiltonian.

The pseudo-vacuum $|0\rangle$ satisfies the following condition

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0. \quad (2.9)$$

where $a(\lambda)$ and $d(\lambda)$ are called vacuum eigenvalues. For Heisenberg spin chain model, the pseudo-vacuum $|0\rangle$ is given by the state where spins are up at all sites.

The Bethe Eigenstates

Consider a state which is obtained by acting B operator on the pseudo-vacuum,

$$|\Psi_N(\lambda)\rangle = \prod_{a=1}^N B(\lambda_a)|0\rangle \quad (2.10)$$

where N is the number of particles or excitations. This state becomes the eigenvector of transfer matrix when spectral parameters, λ_a , satisfy the Bethe ansatz equation, and the eigenvector is called Bethe eigenstate. The dual vector of $|\Psi_N(\lambda)\rangle$ is defined by

$$\langle\Psi_N(\lambda)| = \langle 0| \prod_{a=1}^N C(\lambda_a). \quad (2.11)$$

The vacuum eigenvalues, $a(\lambda)$ and $d(\lambda)$, of inhomogeneous $XXX_{1/2}$ and $XXZ_{1/2}$ model are

$$a(\lambda) = \prod_{j=1}^M \left(\lambda - \nu_j - i\frac{c}{2} \right), \quad d(\lambda) = \prod_{j=1}^M \left(\lambda - \nu_j + i\frac{c}{2} \right), \quad (2.12)$$

and

$$a(\lambda) = \prod_{j=1}^M \sinh(\lambda - \nu_j - i\eta), \quad d(\lambda) = \prod_{j=1}^M \sinh(\lambda - \nu_j + i\eta), \quad (2.13)$$

respectively, and the Bethe ansatz equation is

$$\prod_{j=1}^M \frac{\lambda_a - \nu_j - i\frac{c}{2}}{\lambda_a - \nu_j + i\frac{c}{2}} = e^{i\vartheta} \prod_{\substack{b=1 \\ b \neq a}}^N \frac{\lambda_b - \lambda_a + ic}{\lambda_b - \lambda_a - ic}, \quad (2.14)$$

³For $r \times r$ matrix A and B , $(A \otimes B)_{kl}^{ij} = A_{ij}B_{kl}$, $i, j, k, l = 1, \dots, r$.

$$\prod_{j=1}^M \frac{\sinh(\lambda_a - \nu_j - i\eta)}{\sinh(\lambda_a - \nu_j + i\eta)} = e^{i\vartheta} \prod_{\substack{b=1 \\ b \neq a}}^N \frac{\sinh(\lambda_b - \lambda_a + 2i\eta)}{\sinh(\lambda_b - \lambda_a - 2i\eta)}, \quad (2.15)$$

for the quasi-periodic boundary condition.

The Norm of Bethe Eigenstate for $\text{XXX}_{1/2}$ Model

The norm of Bethe eigenstate [14] is given by

$$\langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle = c^N \prod_{a=1}^N a(\lambda_a) d(\lambda_a) \prod_{a < b} f(\lambda_a, \lambda_b) f(\lambda_b, \lambda_a) \det(\varphi') \quad (2.16)$$

where

$$\varphi'_{ab} = \delta_{ab} \left(i \frac{\partial}{\partial \lambda_a} \log r(\lambda_a) + \sum_{l=1}^N K(\lambda_a, \lambda_l) \right) - K(\lambda_a, \lambda_b) \quad (2.17)$$

$$K(\lambda, \mu) = \frac{2c}{(\lambda - \mu)^2 + c^2}, \quad r(\lambda) = \frac{a(\lambda)}{d(\lambda)} \quad (2.18)$$

For inhomogeneous $\text{XX}X_{\frac{1}{2}}$ spin chain model,

$$\begin{aligned} \varphi'_{ab} = i\delta_{ab} \left[\left(\sum_{l=1}^M \left(\frac{1}{\lambda_a - \nu_l - i\frac{c}{2}} - \frac{1}{\lambda_a - \nu_l + i\frac{c}{2}} \right) + \sum_{s=1}^N \left(\frac{1}{\lambda_s - \lambda_a + ic} + \frac{1}{\lambda_a - \lambda_s + ic} \right) \right) \right. \\ \left. - \left(\frac{1}{\lambda_a - \lambda_b + ic} + \frac{1}{\lambda_b - \lambda_a + ic} \right) \right] =: i\tilde{\varphi}_{ab}, \end{aligned} \quad (2.19)$$

and we obtain

$$\langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle = c^N \prod_{a=1}^N \prod_{j=1}^M \left(\lambda_a - \nu_j - i\frac{c}{2} \right) \left(\lambda_a - \nu_j + i\frac{c}{2} \right) \prod_{a \neq b} \frac{(\lambda_a - \lambda_b + ic)}{(\lambda_b - \lambda_a)} \det(\varphi'). \quad (2.20)$$

Therefore, the inverse of the norm of Bethe eigenstate is given by

$$\begin{aligned} \sum_{(\lambda) \in P_{\text{XXX}}} \langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle^{-1} \\ = \sum_{(\lambda) \in P_{\text{XXX}}} (ic)^{-N} \prod_{a \neq b} \frac{\lambda_a - \lambda_b}{\lambda_a - \lambda_b + ic} \prod_{a=1}^N \prod_{j=1}^M \frac{1}{(\lambda_a - \nu_j - i\frac{c}{2})(\lambda_a - \nu_j + i\frac{c}{2})} \det(\tilde{\varphi}')^{-1} \end{aligned} \quad (2.21)$$

Here P_{XXX} is the set of the independent solutions of $(\lambda) = (\lambda_1, \dots, \lambda_N)$ satisfying the Bethe ansatz (2.15) with the quasi-periodic boundary condition.

The Norm of Bethe Eigenstate for $\text{XXZ}_{1/2}$ Model

The norm of Bethe eigenstates for $\text{XXZ}_{1/2}$ model [14] can be obtained similarly as in the case of $\text{XXX}_{1/2}$ model and it is

$$\begin{aligned} \langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle &= (\sin 2\eta)^N \prod_{a \neq b} \frac{\sinh(\lambda_a - \lambda_b + 2i\eta)}{\sinh(\lambda_a - \lambda_b)} \\ &\times \prod_{a=1}^N \prod_{j=1}^M \sinh(\lambda_a - \nu_j - i\eta) \sinh(\lambda_a - \nu_j + i\eta) \det(\varphi') \end{aligned} \quad (2.22)$$

where

$$\begin{aligned} \varphi'_{ab} &= i\delta_{ab} \left[\sum_{j=1}^M \left(\frac{\cosh(\lambda_a - \nu_j - i\eta)}{\sinh(\lambda_a - \nu_j - i\eta)} - \frac{\cosh(\lambda_a - \nu_j + i\eta)}{\sinh(\lambda_a - \nu_j + i\eta)} \right) + \sum_{e=1}^N \left(\frac{\cosh(\lambda_a - \lambda_e + 2i\eta)}{\sinh(\lambda_a - \lambda_e + 2i\eta)} - \frac{\cosh(\lambda_a - \lambda_e - 2i\eta)}{\sinh(\lambda_a - \lambda_e - 2i\eta)} \right) \right] \\ &- i \left(\frac{\cosh(\lambda_a - \lambda_b + 2i\eta)}{\sinh(\lambda_a - \lambda_b + 2i\eta)} - \frac{\cosh(\lambda_a - \lambda_b - 2i\eta)}{\sinh(\lambda_a - \lambda_b - 2i\eta)} \right) \\ &=: i\tilde{\varphi}'_{ab} \end{aligned} \quad (2.23)$$

In terms of $\tilde{\varphi}'$, the inverse of the norm of Bethe eigenstates is given by

$$\begin{aligned} \sum_{(\lambda) \in P_{\text{XXZ}}} \langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle^{-1} &= \sum_{(\lambda) \in P_{\text{XXZ}}} (i \sin 2\eta)^{-N} \prod_{a \neq b} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b + 2i\eta)} \\ &\times \prod_{a=1}^N \prod_{j=1}^M \frac{1}{\sinh(\lambda_a - \nu_j - i\eta) \sinh(\lambda_a - \nu_j + i\eta)} (\det \tilde{\varphi}')^{-1}, \end{aligned} \quad (2.24)$$

and P_{XXZ} is the set of the independent solutions of $(\lambda) = (\lambda_1, \dots, \lambda_N)$ satisfying the Bethe ansatz equation (2.15) with the quasi-periodic boundary condition.

2.2 Correlation functions in 2d $\mathcal{N} = (2, 2)$ gauge theory and $\text{XXX}_{1/2}$ spin chain model

We consider a topologically twisted $\mathcal{N} = (2, 2)$ $U(N_c)$ gauge theory. The matter chiral multiplets are an adjoint chiral multiplet Φ , N_f fundamental chiral multiplets Q_a^i and anti-fundamental chiral multiplet \tilde{Q}_i^a , $a = 1, \dots, N_c$, $i = 1, \dots, N_f$. The flavor symmetry group is $SU(N_f)_Q \times SU(N_f)_{\tilde{Q}} \times U(1)_D$. The charge assignment is specified in the Table 1.

	$U(N_c)$	$SU(N_f)_Q$	$SU(N_f)_{\tilde{Q}}$	$U(1)_D$	$U(1)_R$
Q	N_c	$\overline{N_f}$	$\mathbf{1}$	$-1/2$	r_1
\tilde{Q}	$\overline{N_c}$	$\mathbf{1}$	N_f	$-1/2$	r_2
Φ	adj	$\mathbf{1}$	$\mathbf{1}$	1	R

Table 1. The matter contents and charge assignment

We denote the mass parameter and flux of Cartan of global symmetries as⁴

$$SU(N_f)_Q : (m_i^y, n_i), \quad SU(N_f)_{\tilde{Q}} : (m_i^{\tilde{y}}, \tilde{n}_i), \quad U(1)_D : (m^z, l). \quad (2.25)$$

The partition function of A-type topologically twisted theory can be calculated by using the formula in [3]. In the following calculation, we turn off the background value of graviphoton associated on S^2 .

The one-loop contributions from chiral, anti-chiral, and adjoint chiral are given by

$$Z_Q^{1\text{-loop}}(k) = \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i^y - \frac{1}{2}m^z)^{r_1 - k_a - 1 + n_i + \frac{1}{2}l} \quad (2.26)$$

$$Z_{\tilde{Q}}^{1\text{-loop}}(k) = \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z)^{r_2 + k_a - 1 - \tilde{n}_i + \frac{1}{2}l} \quad (2.27)$$

$$Z_{\Phi}^{1\text{-loop}}(k) = (m^z)^{N(R-1-l)} \prod_{1 \leq a \neq b \leq N_c} (\sigma_a - \sigma_b + m^z)^{R - (k_a - k_b) - 1 - l} \quad (2.28)$$

and one-loop contribution from vector multiplet is

$$Z_{\text{vector}}^{1\text{-loop}}(k) = (-1)^{\frac{N_c(N_c-1)}{2}} \prod_{1 \leq a < b \leq N_c} (-1)^{k_a - k_b + 1} (\sigma_a - \sigma_b)^2. \quad (2.29)$$

We denote $\sigma = \text{diag}(\sigma_1, \dots, \sigma_{N_c})$ as a constant configuration of the scalar in the vector multiplet. The partition function of A-twisted gauged linear sigma mode on two sphere is given by

$$Z^{2d} = \frac{1}{N_c!} \sum_{\vec{k} \in \mathbb{Z}^{N_c}} q^{\sum_{a=1}^{N_c} k_a} \oint \prod_{a=1}^{N_c} \frac{d\sigma_a}{2\pi i} Z_{\text{total}}^{1\text{-loop}}(k) \quad (2.30)$$

Here the choice of contour is specified by Jeffrey-Kirwan residue prescription. The parameter q is the exponential of the complexified FI-parameter $q := \exp 2\pi i \tau = \exp 2\pi i (\frac{\theta}{2\pi} + i\xi)$ and $Z_{\text{total}}^{1\text{-loop}}(k)$ is

$$Z_{\text{total}}^{1\text{-loop}}(k) = Z_{\text{vector}}^{1\text{-loop}}(k) Z_Q^{1\text{-loop}}(k) Z_{\tilde{Q}}^{1\text{-loop}}(k) Z_{\Phi}^{1\text{-loop}}(k). \quad (2.31)$$

By choosing the covector η , for example, to be $\eta < 0$, the residues are taken from anti-chiral multiplets or part of them with part of factors from adjoint chiral multiplet. Poles from anti-chiral multiplets exist when $k_a < \tilde{n}_i - \frac{1}{2}l - r_2 + 1$. Summing over $k_a < K$ first for sufficiently large positive integer K , the partition function is expressed as

$$\begin{aligned} Z^{2d} = & \frac{(-1)^{\frac{N_c(N_c-1)}{2}}}{N_c!} \oint \left(\prod_{a=1}^{N_c} \frac{d\sigma_a}{2\pi i} \right) \prod_{a=1}^{N_c} \frac{\exp(2\pi i \partial_a \widetilde{\mathcal{W}}_{\text{eff}})^K}{\exp(2\pi i \partial_a \widetilde{\mathcal{W}}_{\text{eff}}) - 1} \prod_{1 \leq a < b \leq N_c} (\sigma_a - \sigma_b)^2 \prod_{1 \leq a \neq b \leq N_c} (\sigma_a - \sigma_b + m^z)^{R-l-1} \\ & \times (m^z)^{N_c(R-l-1)} \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i^y - \frac{1}{2}m^z)^{r_1 + n_i + \frac{1}{2}l - 1} (-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z)^{r_2 - \tilde{n}_i + \frac{1}{2}l - 1} \end{aligned} \quad (2.32)$$

⁴In our notation, the background flux l is an even number.

where $\widetilde{\mathcal{W}}_{\text{eff}}$ is the effective twisted superpotential

$$\begin{aligned} \widetilde{\mathcal{W}}_{\text{eff}} = & \tau \sum_{a=1}^{N_c} \sigma_a - \frac{1}{2} \sum_{1 \leq a < b \leq N_c} (\sigma_a - \sigma_b) \\ & - \frac{1}{2\pi i} \left[\sum_{a=1}^{N_c} \sum_{i=1}^{N_f} (\sigma_a - m_i^y - \frac{1}{2}m^z) (\log(\sigma_a - m_i^y - \frac{1}{2}m^z) - 1) + (-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z) (\log(-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z) - 1) \right. \\ & \left. + \sum_{a,b=1}^{N_c} (\sigma_a - \sigma_b + m^z) (\log(\sigma_a - \sigma_b + m^z) - 1) \right]. \end{aligned} \quad (2.33)$$

and

$$\exp(2\pi i \partial_a \widetilde{\mathcal{W}}_{\text{eff}}) = (-1)^{N_c-1} q \prod_{i=1}^{N_f} \frac{-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z}{\sigma_a - m_i^y - \frac{1}{2}m^z} \prod_{b \neq a} \frac{\sigma_b - \sigma_a + m^z}{\sigma_a - \sigma_b + m^z} \quad (2.34)$$

Due to the factor $\exp(2\pi i \partial_a \widetilde{\mathcal{W}}_{\text{eff}})^K$ with large K in numerator, there are no poles at $-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z = 0$ and $\sigma_a - \sigma_b + m^z = 0$ and only poles at $\exp(2\pi i \partial_a \widetilde{\mathcal{W}}_{\text{eff}}) - 1 = 0$ contribute. Then, dependence on K disappears and we obtain

$$\begin{aligned} Z^{2d} = & (-1)^{\frac{N_c(N_c+1)}{2}} (m^z)^{N_c(R-1-l)} \sum_{\sigma \in P_{2d}} \det(\mathcal{M}^{2d})^{-1} \prod_{a \neq b} (\sigma_a - \sigma_b) (\sigma_a - \sigma_b + m^z)^{R-1-l} \\ & \times \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i^y - \frac{1}{2}m^z)^{r_1-1+n_i+\frac{1}{2}l} (-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z)^{r_2-1-\tilde{n}_i+\frac{1}{2}l} \end{aligned} \quad (2.35)$$

where

$$P_{2d} := \{(\sigma_1, \dots, \sigma_{N_c}) \mid \exp(2\pi i \partial_{\sigma_a} \widetilde{\mathcal{W}}_{\text{eff}}) = 1 \text{ for all } a = 1, \dots, N_c\} \quad (2.36)$$

$$\mathcal{M}_{ab}^{2d} := (-2\pi i) \partial_{\sigma_a} \partial_{\sigma_b} \widetilde{\mathcal{W}}_{\text{eff}} \quad (2.37)$$

and

$$-2\pi i \partial_{\sigma_a} \partial_{\sigma_b} \widetilde{\mathcal{W}}_{\text{eff}} = \delta_{ab} \left(\frac{1}{\sigma_a - m_i^y - \frac{1}{2}m^z} + \frac{1}{-\sigma_a + m_i^{\tilde{y}} - \frac{1}{2}m^z} + \sum_{l=1}^{N_c} (\mathcal{S}_{lb} - \mathcal{S}_{bl}) \right) - \mathcal{S}_{ab} - \mathcal{S}_{ba} \quad (2.38)$$

with

$$\mathcal{S}_{kl} = \frac{1}{\sigma_k - \sigma_l + m^z} \quad (2.39)$$

In P_{2d} , we removed solutions which are same up to the permutations of $(\sigma_1, \dots, \sigma_{N_c})$. And the condition for supersymmetric vacua, $\exp(2\pi i \partial_{\sigma_a} \widetilde{\mathcal{W}}_{\text{eff}}) = 1$, is given by

$$\prod_{i=1}^{N_f} \frac{(\sigma_a - m_i^y - \frac{1}{2}m^z)}{(\sigma_a - m_i^{\tilde{y}} + \frac{1}{2}m^z)} = (-1)^{N_f} e^{2\pi i \tau} \prod_{b \neq a}^{N_c} \frac{\sigma_b - \sigma_a + m^z}{\sigma_b - \sigma_a - m^z}. \quad (2.40)$$

Comparison and Matching

The following identification

$$m^z = ic, \quad m^y = m^{\tilde{y}} = \nu, \quad N_c = N, \quad N_f = M, \quad (-1)^{N_f} q = e^{i\vartheta} \quad (2.41)$$

give the agreement between Bethe ansatz equation (2.14) for $\text{XXX}_{1/2}$ spin chain model and the condition for supersymmetric vacua (2.40) of 2d $\mathcal{N} = (2, 2)$ gauge theory. Moreover, with identification

$$r_1 + n_i + \frac{l}{2} = 0, \quad r_2 - \tilde{n}_i + \frac{l}{2} = 0, \quad R - l = 0, \quad (2.42)$$

the partition function of A-twisted 2d $\mathcal{N} = (2, 2)$ gauge theory⁵ and the inverse of the norm of the Bethe eigenstates (2.21) agree;

$$Z^{2d} = \sum_{(\lambda) \in P_{\text{XXX}}} \langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle^{-1} \quad (2.44)$$

This type of relation was first studied for $U(N)/U(N)$ gauged WZW model on genus- g Riemann surfaces Σ_g in [15] where the corresponding integrable model is phase model. This correspondence was generalized to one-parameter deformation $U(N)/U(N)$ gauged WZW model and q -boson model in [16]. The relation between the deformation of gauged WZW model and Chern-Simons matter theories on $S^1 \times \Sigma_g$ was discussed in [17, 18]. See also [19].

Correlation Functions and Conserved Charges

We have identified the partition functions and the norm of the inverse of Bethe eigenstates. We can also consider the correlation functions of A-twisted 2d $\mathcal{N} = (2, 2)^*$ theory in the context of Gauge-Bethe correspondence.

In A-twisted 2d $\mathcal{N} = (2, 2)$ theory, the correlation functions of the gauge invariant operators $\mathcal{O}(\sigma)$ are given by

$$\langle \mathcal{O}(\sigma) \rangle = \frac{1}{N_c!} \sum_{\vec{k} \in \mathbb{Z}^{N_c}} (-1)^{(N_c-1) \sum_{a=1}^{N_c} k_a} q^{\sum_{a=1}^{N_c} k_a} \oint \prod_{a=1}^{N_c} \frac{d\sigma_a}{2\pi i} \mathcal{O}(\sigma) Z_{\text{total}}^{1\text{-loop}}(k). \quad (2.45)$$

⁵For example, we can choose all the background fluxes and R -charges to be zero. However the canonical assignment of R -charge is not allowed if we want to match the A-twisted partition function and the inverse of the norm of the Bethe eigenstates. Indeed if we sum three conditions in (2.42), we obtain

$$N_f(r_1 + r_2 + R) + \sum_{i=1}^{N_f} n_i - \sum_{i=1}^{N_f} \tilde{n}_i = 0. \quad (2.43)$$

However, as the flavor symmetries are $SU(N_f)$ instead of $U(N_f)$, we have $r_1 + r_2 + R = 0$. Therefore, the canonical assignment of the R -charge, $r_1 = r_2 = 0$ and $R = 2$, is not allowed for the match with the inverse of the norm. Also note that, given same matter contents, Bethe ansatz equation are same whatever R -charges and background magnetic fluxes are.

This can also be written as

$$\langle \mathcal{O}(\sigma) \rangle = \sum_{\sigma \in P_{2d}} \mathcal{O}(\sigma) \frac{Z_{\text{total}}^{1\text{-loop}}(k=0)}{\det \mathcal{M}^{2d}} \quad (2.46)$$

The operator $\mathcal{O}(\sigma)$ is provided by gauge invariant polynomials of Cartan of the scalar component σ of vector multiplet, which is a symmetric function of σ_a , $a = 1, \dots, N_c$. Thus it can be written in terms of the elementary symmetric polynomials. We denote the polynomial $Q(x)$ as

$$Q(x) = \prod_{a=1}^{N_c} (x - \sigma_a), \quad (2.47)$$

then the coefficients of x^{N_c-l} gives l -th elementary symmetric polynomial of σ_a .

Meanwhile, in integrable models there is a fundamental quantity known as Baxter Q -operator $\mathbf{Q}(x)$ whose eigenvalue is actually (2.47) with N_c identified with the number of particles N and σ_a identified with spectral parameters λ_a . Thus, we see that the expectation value of Baxter Q -operator provides the generating function of correlation function of gauge invariant operators in 2d $\mathcal{N} = (2, 2)^*$ theory, *i.e.*

$$\begin{aligned} \sum_{(\lambda) \in P_{\text{XXX}}} \frac{\langle \Psi_N(\lambda) | \mathbf{Q}(x) | \Psi_N(\lambda) \rangle}{\langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle^2} &= \sum_{\lambda_a \in P_{\text{XXX}}} (ic)^{-N} Q(x) \det^{-1}(\tilde{\varphi}') \\ &\times \prod_{a \neq b} \frac{\lambda_a - \lambda_b}{\lambda_a - \lambda_b + ic} \prod_{a=1}^N \prod_{j=1}^M \frac{1}{(\lambda_a - \nu_j - i\frac{c}{2})(\lambda_a - \nu_j + i\frac{c}{2})}. \end{aligned} \quad (2.48)$$

The eigenvalue of transfer matrix $\tau(\mu)$ for $\text{XXX}_{1/2}$ model is given by

$$\theta(\mu, \{\lambda_a\}) = \mathbf{a}(\mu) \prod_{a=1}^N f(\mu, \lambda_a) + e^{i\vartheta} \mathbf{d}(\mu) \prod_{a=1}^N f(\lambda_a, \mu). \quad (2.49)$$

Therefore, the eigenvalue $\theta(\mu, \{\lambda_a\})$ is generating function of symmetric polynomial of λ_a . As discussed above, the eigenvalue of transfer matrix is actually generating function of mutually commuting conserved charges (or Hamiltonians). Accordingly, we can identify the expectation value of conserved charges of $\text{XXX}_{1/2}$ spin chain model with the twisted GLSM correlators with appropriate coefficients.

2.3 Correlation functions in 3d $\mathcal{N} = 2$ theory and $\text{XXZ}_{1/2}$ spin chain model

We consider a topologically twisted 3d $\mathcal{N} = 2$ $U(N_c)$ gauge theory with an adjoint chiral multiplet Φ and N_f chiral and anti-chiral multiplet Q^{ai} , \tilde{Q}_{ai} , $a = 1, \dots, N_c$, $i = 1, \dots, N_f$, respectively, where we use same notation as 2d case. There are flavor symmetries $SU(N_f)_Q$, $SU(N_f)_{\tilde{Q}}$ and $U(1)_D$. In addition, there is a $U(1)_T$ topological symmetry in three dimensions. The matter contents and charge assignment are specified in Table 2.

	$U(N_c)$	$SU(N_f)_Q$	$SU(N_f)_{\tilde{Q}}$	$U(1)_D$	$U(1)_T$	$U(1)_R$
Q	N_c	\overline{N}_f	$\mathbf{1}$	$-1/2$	0	r_1
\tilde{Q}	\overline{N}_c	$\mathbf{1}$	N_f	$-1/2$	0	r_2
Φ	adj	$\mathbf{1}$	$\mathbf{1}$	1	0	R

Table 2. Matter contents of 3d $\mathcal{N} = 2$ theory

We denote fugacities and magnetic fluxes of the Cartan part of global symmetries as follows;

$$SU(N_f)_Q : (y_i, n_i), \quad SU(N_f)_{\tilde{Q}} : (\tilde{y}_i, \tilde{n}_i), \quad U(1)_D : (z, l), \quad U(1)_T : (\zeta, u) \quad (2.50)$$

Then the topologically twisted index of the 3d $\mathcal{N} = 2$ theory is given by

$$\begin{aligned} Z^{3d} = & \frac{1}{N_c!} \sum_{\vec{m} \in \mathbb{Z}^{N_c}} \int \prod_{a=1}^{N_c} \frac{dx_a}{2\pi i x_a} (-1)^{(N_c-1) \sum_{a=1}^{N_c} m_a} \prod_{a \neq b}^{N_c} \left(1 - \frac{x_a}{x_b}\right) \prod_{a,b=1}^{N_c} \left(\frac{x_a^{1/2} x_b^{-1/2} z^{1/2}}{1 - x_a x_b^{-1} z} \right)^{m_a - m_b + l - R + 1} \\ & \times \prod_{a=1}^{N_c} \zeta^{m_a} \prod_{i=1}^{N_f} \left(\frac{x_a^{1/2} y_i^{-1/2} z^{-1/4}}{1 - x_a y_i^{-1} z^{-1/2}} \right)^{m_a - n_i - \frac{l}{2} - r_1 + 1} \left(\frac{x_a^{-1/2} \tilde{y}_i^{1/2} z^{-1/4}}{1 - x_a^{-1} \tilde{y}_i z^{-1/2}} \right)^{-m_a + \tilde{n}_i - \frac{l}{2} - r_2 + 1} \end{aligned} \quad (2.51)$$

Here x_a is a constant value of Wilson loop for a -th diagonal $U(1)$ of gauge group $U(N_c)$. We take, for example, $\eta = (-1, -1, \dots, -1)$ to choose a contour so that it picks the poles from anti-fundamental chiral multiplets or from part of them with part of factors from adjoint chiral. Poles exist for $m_a < \tilde{n}_i - \frac{l}{2} - r_2 + 1$, and we resum over $m_i < K$ for sufficiently large positive integer K . With $f_i = -n_i - \frac{l}{2} - r_1 + 1$, $\tilde{f}_i = \tilde{n}_i - \frac{l}{2} - r_2 + 1$, and $h = l - R + 1$, above expression can be organized to

$$\begin{aligned} Z^{3d} = & \frac{(-1)^{\frac{N_c(N_c+1)}{2}} z^{hN_c^2/2}}{N_c! (1-z)^{hN_c}} \sum_{\vec{m} \in \mathbb{Z}^{N_c}} \int \prod_{a=1}^{N_c} \frac{dx_a}{2\pi i x_a} \prod_{b \neq a}^{N_c} \frac{1 - \frac{x_a}{x_b}}{\left(1 - \frac{x_a}{x_b} z\right)^h} \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} \frac{(x_a y_i^{-1} z^{-1/2})^{f_i/2} (x_a^{-1} \tilde{y}_i z^{-1/2})^{\tilde{f}_i/2}}{(1 - x_a y_i^{-1} z^{-1/2})^{f_i} (1 - x_a^{-1} \tilde{y}_i z^{-1/2})^{\tilde{f}_i}} \\ & \times \prod_{a=1}^{N_c} \zeta^{m_a} (-1)^{(N_c-1)m_a} \prod_{\substack{b=1 \\ b \neq a}}^{N_c} \left(\frac{x_a - x_b z}{x_b - x_a z} \right)^{m_a} \prod_{i=1}^{N_f} \prod_{a=1}^{N_c} \left(\frac{x_a y_i^{-1/2} \tilde{y}_i^{-1/2}}{(1 - x_a y_i^{-1} z^{-1/2})(1 - x_a^{-1} \tilde{y}_i z^{-1/2})^{-1}} \right)^{m_a} \end{aligned} \quad (2.52)$$

and summing over all fluxes for $m_i < K$, we get

$$\begin{aligned} Z^{3d} = & \frac{(-1)^{\frac{N_c(N_c+1)}{2}} z^{hN_c^2/2}}{N_c! (1-z)^{hN_c}} \int \prod_{a=1}^{N_c} \frac{dx_a}{2\pi i x_a} \prod_{b \neq a}^{N_c} \frac{1 - \frac{x_a}{x_b}}{\left(1 - \frac{x_a}{x_b} z\right)^h} \\ & \times \prod_{a=1}^{N_c} \prod_{i=1}^{N_f} \left(\frac{(x_a y_i^{-1} z^{-1/2})^{1/2}}{1 - x_a y_i z^{-1/2}} \right)^{f_i} \left(\frac{(x_a^{-1} \tilde{y}_i z^{-1/2})^{1/2}}{1 - x_a^{-1} \tilde{y}_i z^{-1/2}} \right)^{\tilde{f}_i} \frac{(\zeta e^{iB_a(x)})^K}{\zeta e^{iB_a(x)} - 1} \end{aligned} \quad (2.53)$$

and $B_a(x)$ is

$$\exp(iB_a(x)) := \prod_{\substack{b=1 \\ b \neq a}}^{N_c} \left(\frac{x_a - x_b z}{x_b - x_a z} \right) \prod_{i=1}^{N_f} \left(\frac{x_a y_i^{-1/2} \tilde{y}_i^{-1/2}}{(1 - x_a y_i^{-1} z^{-1/2})(1 - x_a^{-1} \tilde{y}_i z^{-1/2}) - 1} \right) \quad (2.54)$$

Due to $(\zeta e^{iB_a(x)})^K$ with large K in numerator, poles at $x_a = 0$, $1 - x_a^{-1} \tilde{y}_i z^{-1/2} = 0$, and $x_a - x_b z = 0$ are not available and only relevant poles come from $\zeta e^{iB_a(x)} = 1$ for all a . We denote the solution for this equation as

$$P_{3d} = \{(x_1, \dots, x_{N_c}) \mid \zeta e^{iB_a(x)} = 1, \text{ for all } a = 1, 2, \dots, N_c\}. \quad (2.55)$$

where we removed solutions which are same up to permutation of (x_1, \dots, x_{N_c}) .

With

$$x_a = e^{2\lambda_a}, \quad y_j = \tilde{y}_j = e^{2\nu_j}, \quad z = e^{4i\eta}, \quad (-1)^{N_f} \zeta = e^{i\vartheta}. \quad (2.56)$$

the contour integral gives

$$\begin{aligned} Z^{3d} = & \frac{1}{(2i \sin 2\eta)^{N_c}} \sum_{\lambda_a \in P_{3d}} (\det \mathcal{M}^{3d})^{-1} \prod_{a < b}^{N_c} (x_b x_a)^{h-1} \prod_{a \neq b}^{N_c} \frac{\sinh(\lambda_a - \lambda_b)}{(\sinh(\lambda_a - \lambda_b - 2i\eta))^h} \\ & \times \left(\prod_{a=1}^{N_c} \prod_{i=1}^{N_f} \left(\frac{1}{2 \sinh(\lambda_a - \nu_i - i\eta)} \right)^{f_i} \left(-\frac{1}{2 \sinh(\lambda_a - \nu_i + i\eta)} \right)^{\tilde{f}_i} \right) \end{aligned} \quad (2.57)$$

where we used

$$\frac{\partial e^{iB_a(x)}}{\partial x_b} = \frac{e^{iB_a(x)}}{x_b} \frac{\partial B_a(x)}{\partial \lambda_b} \quad (2.58)$$

with

$$\begin{aligned} \mathcal{M}_{ab}^{3d} := & -\frac{\partial iB_a(x)}{\partial \lambda_b} = \\ \delta_{ab} \left[& \sum_{j=1}^{N_f} \left(\frac{\cosh(\lambda_a - \nu_j - i\eta)}{\sinh(\lambda_a - \nu_j - i\eta)} - \frac{\cosh(\lambda_a - \nu_j + i\eta)}{\sinh(\lambda_a - \nu_j + i\eta)} \right) + \sum_{e=1}^{N_c} \left(\frac{\cosh(\lambda_a - \lambda_e + 2i\eta)}{\sinh(\lambda_a - \lambda_e + 2i\eta)} - \frac{\cosh(\lambda_a - \lambda_e - 2i\eta)}{\sinh(\lambda_a - \lambda_e - 2i\eta)} \right) \right] \\ & - \left(\frac{\cosh(\lambda_a - \lambda_b + 2i\eta)}{\sinh(\lambda_a - \lambda_b + 2i\eta)} - \frac{\cosh(\lambda_a - \lambda_b - 2i\eta)}{\sinh(\lambda_a - \lambda_b - 2i\eta)} \right) \end{aligned} \quad (2.59)$$

Also, upon (2.56), the condition for supersymmetric vacua, $\zeta e^{iB_a(x)} = 1$, *i.e.*

$$\prod_{j=1}^{N_f} \frac{\sinh(\lambda_a - \nu_j - i\eta)}{\sinh(\lambda_a - \nu_j + i\eta)} = e^{i\vartheta} \prod_{b \neq a}^{N_c} \frac{\sinh(\lambda_b - \lambda_a + 2i\eta)}{\sinh(\lambda_b - \lambda_a - 2i\eta)} \quad (2.60)$$

is exactly same as the Bethe ansatz for XXZ_{1/2} spin chain (2.15) with $N_c = N$, $N_f = M$.

If we choose R -charges and magnetic fluxes in such a way that

$$r_1 + n_i + \frac{l}{2} = 0, \quad r_2 - \tilde{n}_i + \frac{l}{2} = 0, \quad R - l = 0 \quad (2.61)$$

holds, then 3d topologically twisted index (2.57) and the inverse of the norm of Bethe eigenstate of $\text{XXZ}_{1/2}$ spin chain model agree

$$Z^{3d} = \sum_{(\lambda) \in P_{\text{XXZ}}} \langle \Psi_N(\lambda) | \Psi_N(\lambda) \rangle^{-1} \quad (2.62)$$

up to overall constants.

We can also consider correlation functions and conserved charges in 3d $\mathcal{N} = 2$ theory and $\text{XXZ}_{1/2}$ spin chain model as in section 2.3. The eigenvalue $Q(u)$ of the Baxter Q -operator $\mathbf{Q}(u)$ in $\text{XXZ}_{1/2}$ model is given by

$$Q(u) = \prod_{a=1}^N \sinh(u - \lambda_a),$$

or $\frac{1}{2^N} e^{-Nu - \sum_{a=1}^N \lambda_a} \prod_{a=1}^N (e^{2u} - e^{2\lambda_a})$. Meanwhile, the Wilson loop in 3d $\mathcal{N} = 2$ theories is given by Schur polynomial

$$W_{\mathcal{R}}(x) = s_Y(x_1, \dots, x_{N_c})$$

where Y is a Young tableaux for the representation \mathcal{R} of $U(N_c)$. When \mathcal{R} is totally anti-symmetric representation $Y = 1^r$, $r = 1, \dots, N_c$, Schur polynomial is given by elementary symmetric polynomial, $s_{1^r}(x_1, \dots, x_{N_c}) = e_r(x_1, \dots, x_{N_c})$. Therefore, upon (2.56), the expectation value of Wilson loop operators is proportional to the coefficient of the eigenvalue of the Baxter Q -operator.

Also, as the eigenvalue of the transfer matrix $\tau(\mu)$ for $\text{XXZ}_{1/2}$ model is also (2.49) with (2.13), we can identify the expectation value of conserved charges of $\text{XXZ}_{1/2}$ model with the expectation value of Wilson loop with appropriate coefficients.

3 Equivariant Quantum Cohomology, GLSM, and Integrable Model

In the previous section, we studied the relation between an A-twisted $\mathcal{N} = (2, 2)$ GLSM and correlation functions of $\text{XXX}_{1/2}$ spin chain. It is known that correlation functions of an A-twisted $\mathcal{N} = (2, 2)$ GLSM with a particular choice of R-charge give the integration of quantum cohomology classes of Fano manifolds, which are Higgs branch vacua of the GLSM [20] [add references]. We expect that a similar story holds for $\mathcal{N} = (4, 4)$ GLSM where Higgs branch vacua is now hyperKähler manifold. We turn on all the possible twisted mass parameters including the one associated to $\mathcal{N} = (2, 2)$ adjoint chiral multiplet⁶. In this section, we consider correlation functions of A-twisted $\mathcal{N} = (2, 2)^*$ GLSM on S^2 and study its relation to equivariant quantum cohomology of cotangent bundle of Grassmannians.

⁶Since the $\mathcal{N} = (2, 2)$ adjoint chiral multiplet have flat direction, we have to turn on the $U(1)_D$ twisted mass.

3.1 Equivariant quantum cohomology and Equivariant Integration

Firstly, we summarize the equivariant quantum cohomology of the cotangent bundle of Grassmannians $T^*\text{Gr}(r, n)$ [9]. A Grassmannian $\text{Gr}(r, n)$ are specified by the chains of subspaces,

$$0 = F_0 \subset F_1 \subset F_2 = \mathbb{C}^n \quad (3.1)$$

with $\dim F_1 = r$. It is convenient to define $(\lambda_1, \lambda_2) := (r, n - r)$. We would like to consider the cotangent bundle $T^*\text{Gr}(r, n)$ of Grassmannian $\text{Gr}(r, n)$.

There is a torus action $(\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$ on \mathbb{C}^n , accordingly on $\text{Gr}(r, n)$. In addition, there is also \mathbb{C}^* action on the fiber direction of $T^*\text{Gr}(r, n)$. With these actions, one can consider $GL_n(\mathbb{C}) \times \mathbb{C}^*$ equivariant cohomology ring. We denote $\Gamma_i = \{\gamma_{i,1}, \dots, \gamma_{i,\lambda_i}\}$ with $i = 1, 2$ as the set of Chern roots of bundles on $\text{Gr}(r, n)$ with fiber F_i/F_{i-1} , $\mathbf{z} = \{z_1; \dots; z_n\}$ as Chern roots corresponding to each factors of $(\mathbb{C}^*)^n$ action, and h as the Chern root corresponding to \mathbb{C}^* action. Then the equivariant cohomology ring is given by

$$H_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C}) = \mathbb{C}[\mathbf{z}, \Gamma, h]^{S_n \times S_{\lambda_1} \times S_{\lambda_2}} / \mathcal{I} \quad (3.2)$$

where S_n , S_{λ_1} and S_{λ_2} denote the symmetrization of variables $\{z_1, \dots, z_n\}$, $\{\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}\}$ and $\{\gamma_{2,1}, \dots, \gamma_{2,\lambda_2}\}$, respectively. The ideal \mathcal{I} is generated by the n coefficients of a degree $n - 1$ polynomial of u :

$$\prod_{a=1}^2 \prod_{b=1}^{\lambda_a} (u - \gamma_{a,b}) - \prod_{i=1}^n (u - z_i). \quad (3.3)$$

The equivariant quantum cohomology ring of cotangent bundle of Grassmannian is given by

$$QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C}) = \mathbb{C}[\mathbf{z}, \Gamma, h]^{S_n \times S_{\lambda_1} \times S_{\lambda_2}} \otimes \mathbb{C}[[\mathbf{q}]] / \mathcal{I}_{\mathbf{q}} \quad (3.4)$$

where $\mathbb{C}[[\mathbf{q}]]$ is the ring of formal series of \mathbf{q} , which is the quantum parameter. The ideal $\mathcal{I}_{\mathbf{q}}$ is generated by the n coefficients p_l defined by

$$\begin{aligned} \sum_{l=1}^n p_l(\mathbf{z}, \Gamma, h, \mathbf{q}) u^{n-l} &:= \prod_{a=1}^2 \prod_{b=1}^{\lambda_a} (u - \gamma_{a,b}) \\ &- \mathbf{q} \prod_{a=1}^{\lambda_1} (u - \gamma_{1,a} - h) \prod_{b=1}^{\lambda_2} (u - \gamma_{2,b} + h) - (1 - \mathbf{q}) \prod_{i=1}^n (u - z_i) \end{aligned} \quad (3.5)$$

The coefficients p_l are degree l polynomials of each Γ and \mathbf{z} and invariant under the action of $S_n \times S_{\lambda_1} \times S_{\lambda_2}$. Meanwhile, the authors of [21] constructed the Yangian acting on equivariant cohomology and identified equivariant quantum cohomology ring as Bethe subalgebra. The cotangent bundle of Grassmannian is a typical example of [21].

The equivariant integration of a cohomology class $[f(\Gamma, \mathbf{z}, h)] \in H_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C})$ is calculated by the formula

$$\int_{T^*\text{Gr}(r, n)} [f] = (-1)^{\lambda_1 \lambda_2} \sum_{I_r \subset I} \prod_{i \in I_r} \prod_{j \in I_{n-r}} \frac{f(\mathbf{z}_I, \mathbf{z}; h)}{(z_i - z_j)(z_i - z_j + h)} \quad (3.6)$$

where I_r is a subsets of $I = \{1, \dots, n\}$ with $|I_r| = r$ and I_{n-r} is the compliment of I_r in I . The factor $f(\mathbf{z}_I, \mathbf{z}; h)$ in numerator is defined by the substitution $\Gamma = (\Gamma_1, \Gamma_2) \rightarrow (\mathbf{z}_{I_r}, \mathbf{z}_{I_{n-r}})$ in $f(\Gamma, \mathbf{z}, h)$. Summation $\sum_{I_r \subset I}$ in (3.6) runs for all the possible subset in I with fixed r .

As will be discussed with explicit examples in section 3.2, by considering the case of Fano target space and also from the comparison with the GLSM results, we expect that the equivariant integration of the element $[f(\Gamma, \mathbf{z}, h; q)]$ of equivariant quantum cohomology ring $QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C})$ is also calculated by using the formula (3.6).⁷ More specifically, given a ring element we reduce the degree of the ring element by using the ideal \mathcal{I}_q whenever it is possible and then apply the formula (3.6) to the resulting ring element, which generically depends on the parameter q . The GLSM correlation function of the operator corresponding to a given original ring element is expected to match with the equivariant integration obtained in a way we have just described.

3.2 Correlation functions of A-twisted GLSM and equivariant integration of equivariant quantum cohomology

We study the relation between correlation functions of A-twisted 2d $\mathcal{N} = (2, 2)^*$ GLSM and equivariant integration for quantum cohomology.

The gauge group and the matter content are same as section 2.2, but we choose different R -charge from the previous case in such a way that we have now the superpotential

$$W_{\tilde{Q}\Phi Q} = \sum_{a,b=1}^{N_c} \sum_{i=1}^{N_f} \tilde{Q}_i^a \Phi_a^b Q_b^i. \quad (3.7)$$

Then the Higgs branch vacua is $T^*\text{Gr}(N_c, N_f)$ in positive FI-parameter regions. The superpotential breaks $SU(N_f)_Q \times SU(N_f)_{\tilde{Q}} \times U(1)_D$ to $SU(N_f) \times U(1)_D$. We turn off all the background flux for flavor symmetry group. We denote the twisted mass parameters for $SU(N_f)$ flavor symmetry by m_i and the twisted masses parameter for $U(1)_D$ flavor symmetry by m^z .

	$U(N_c)$	$SU(N_f)$	$U(1)_D$	$U(1)_R$
Q	N_c	\bar{N}_f	$-1/2$	0
\tilde{Q}	\bar{N}_c	N_f	$-1/2$	0
Φ	adj	$\mathbf{1}$	1	2

Table 3. The charge assignment of GLSM

⁷As we will see, we expect that it works for $T^*\text{Gr}(r, n)$ when $r \leq n-r$, and we use ideal when calculating the case $r \geq n-r$.

The correlation function of gauge invariant operator $\mathcal{O}(\sigma)$ constructed from $\sigma = \text{diag}(\sigma_1, \dots, \sigma_{N_c})$ is

$$\begin{aligned} \langle \mathcal{O}(\sigma) \rangle_{\text{A-twist}}^{N_c, N_f} &= \frac{(-1)^{N_*}}{N_c!} \sum_{a=1}^{N_c} \sum_{k_a=0}^{\infty} ((-1)^{N_c-1} q)^{\sum_{a=1}^{N_c} k_a} \oint \prod_{a=1}^{N_c} \frac{d\sigma_a}{2\pi i} \mathcal{O}(\sigma) \\ &\times \frac{\prod_{1 \leq a \neq b \leq N_c} (\sigma_a - \sigma_b)}{\prod_{a,b=1}^{N_c} (\sigma_a - \sigma_b + m^z)^{k_a - k_b - 1}} \prod_{i=1}^{N_f} \prod_{a=1}^{N_c} \frac{(-\sigma_a + m_i - \frac{m^z}{2})^{k_a - 1}}{(\sigma_a - m_i - \frac{m^z}{2})^{k_a + 1}} \end{aligned} \quad (3.8)$$

Here we take the charge vector in Jeffrey–Kirwan residues as $\text{Re } q < 1$. Then the residue is evaluated at the poles $(\sigma_a - m_i - \frac{m^z}{2})^{-(k_a+1)}$ and it is easy to show that poles coming from $(\sigma_a - \sigma_b + m^z)^{-(k_a - k_b - 1)}$ do not contribute to the residues. The overall sign ambiguity will be fixed below.

Bethe ansatz equation and matching of parameters

Before considering correlation functions, let us see how the twisted chiral ring relation (2.40) and the Bethe ansatz equation, $\exp\left(2\pi i \frac{\partial \widetilde{\mathcal{W}}_{\text{eff}}}{\partial \sigma_a}\right) = 1$, arise from the ideal of equivariant quantum cohomology (3.5) to identify the parameters. By substituting $u = \gamma_{1,c}$ and $\gamma_{1,c} + h$ into

$$0 = \sum_{l=1}^n p_l(\mathbf{z}, \Gamma, h, \mathbf{q}) u^{n-l}, \quad (3.9)$$

we obtain two equations

$$-\mathbf{q} \prod_{a=1}^{\lambda_1} (\gamma_{1,c} - \gamma_{1,a} - h) \prod_{b=1}^{\lambda_2} (\gamma_{1,c} - \gamma_{2,b} + h) = (1 - \mathbf{q}) \prod_{i=1}^n (\gamma_{1,c} - z_i), \quad (3.10)$$

$$\prod_{a=1}^{\lambda_1} (\gamma_{1,c} - \gamma_{1,a} + h) \prod_{b=1}^{\lambda_2} (\gamma_{1,c} - \gamma_{2,b} + h) = (1 - \mathbf{q}) \prod_{i=1}^n (\gamma_{1,c} - z_i + h) \quad (3.11)$$

Dividing (3.10) by (3.11), we get

$$\mathbf{q} \prod_{\substack{a=1 \\ a \neq c}}^{\lambda_1} \frac{\gamma_{1,c} - \gamma_{1,a} + h}{\gamma_{1,c} - \gamma_{1,a} - h} = \prod_{i=1}^n \frac{\gamma_{1,c} - z_i}{\gamma_{1,c} - z_i + h} \quad (3.12)$$

Then the identification

$$r = N_c, \quad n = N_f, \quad \gamma_{1,a} = \sigma_a, \quad z_i = m_i + \frac{m^z}{2}, \quad h = m^z, \quad \mathbf{q} = (-1)^{N_f} q. \quad (3.13)$$

gives the condition for supersymmetric vacua (2.40) for $\mathcal{N} = (2, 2)^* U(N_c)$ GLSM with N_f fundamental hypermultiplets.

Quantum cohomology of \mathbb{CP}^{n-1} and correlation functions of A-twisted GLSM

Let us briefly recall the well known relation between $\mathcal{N} = (2, 2)$ U(1) GLSM with n charge +1 chiral multiplets and quantum cohomology of \mathbb{CP}^{n-1} . This GLSM flows to $\mathcal{N} = (2, 2)$ non-linear sigma model with target space \mathbb{CP}^{n-1} [22] [cite references]. The quantum cohomology of \mathbb{CP}^{n-1} is given by

$$QH^*(\mathbb{CP}^{n-1}; \mathbb{C}) \simeq \mathbb{C}[\mathbf{q}, \gamma_{1,1}] / (\gamma_{1,1}^n - \mathbf{q}) \quad (3.14)$$

The equivariant integration of $\gamma_{1,1}^a \in QH^*(\mathbb{CP}^{n-1}; \mathbb{C})$, which we denote as $\langle \gamma_{1,1}^a \rangle_{\mathbb{CP}^{n-1}}$, is written as follows. If $a < n$, $\langle \gamma_{1,1}^l \rangle_{\mathbb{CP}^{n-1}}$ is same as integration of cohomology class $\gamma_{1,1}^a \in H^*(\mathbb{CP}^{n-1}; \mathbb{C})$ and is given by

$$\langle \gamma_{1,1}^a \rangle_{\mathbb{CP}^{n-1}} = \int_{\mathbb{CP}^{n-1}} \gamma_{1,1}^a = \begin{cases} 1 & a = n-1, \\ 0 & a < n-1. \end{cases} \quad (3.15)$$

If $l = mn + a$ with $a < n$, we reduce the degree by using the relation $\gamma_{1,1}^n - \mathbf{q} = 0$ to $\gamma_{1,1}^{an+l} = \mathbf{q}^a \gamma_{1,1}^l$ and obtain

$$\langle \gamma_{1,1}^{an+l} \rangle_{\mathbb{CP}^{n-1}} = \mathbf{q}^a \int_{\mathbb{CP}^{n-1}} \gamma_{1,1}^l = \begin{cases} \mathbf{q}^a & l = n-1, \\ 0 & l < n-1. \end{cases} \quad (3.16)$$

On the other hand, the expectation value of σ^m is evaluated by supersymmetric localization as

$$\langle \sigma^m \rangle_{\text{A-twist}} = \sum_{k=0}^{\infty} q^k \oint_{\sigma=0} \frac{d\sigma}{2\pi i} \sigma^{m-n(k+1)}, \quad (3.17)$$

which gives

$$\langle \sigma^{an+l} \rangle_{\text{A-twist}} = \begin{cases} q^a & l = n-1, \\ 0 & l < n-1. \end{cases} \quad (3.18)$$

Therefore we have

$$\langle \gamma_{1,1}^m \rangle_{\mathbb{CP}^{n-1}} = \langle \sigma^m \rangle_{\text{A-twist}}, \quad \text{with } q = \mathbf{q}. \quad (3.19)$$

We expect that similar way of calculation for equivariant integration as above also works for our case.

3.2.1 $T^*\mathbb{CP}^{n-1}$

We would like to relate the expectation value of σ^l in GLSM to the equivariant integration of quantum cohomology when target space is $T^*\mathbb{CP}^{n-1}$.

$T^*\mathbb{CP}^1$

From (3.12), the following relation holds

$$\gamma_{1,1}^2 = (z_1 + z_2)\gamma_{1,1} + \frac{2h\mathbf{q}}{1-\mathbf{q}}\gamma_{1,1} + \frac{h\mathbf{q}(h - z_1 - z_2) + \mathbf{q}z_1z_2}{1-\mathbf{q}}. \quad (3.20)$$

This relation is same as chiral ring relation via (3.13). By using (3.20), $\gamma_{1,1}^l$ can be uniquely expressed as

$$\gamma_{1,1}^l = A_l^{(1)}(z, h, \mathbf{q})\gamma_{1,1} + A_l^{(0)}(z, h, \mathbf{q}) \quad (3.21)$$

From (3.21), we expect that equivariant integration of $\gamma_{1,1}^l$ on equivariant quantum cohomology is given by

$$\langle \gamma_{1,1}^l \rangle_{T^*\mathbb{CP}^1} = A_l^{(1)}(z, h, \mathbf{q}) \int_{T^*\mathbb{CP}^1} [\gamma_{1,1}] + A_l^{(0)}(z, h, \mathbf{q}) \int_{T^*\mathbb{CP}^1} [1]. \quad (3.22)$$

Then $\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=2}$ is expected to be related to equivariant integral on $T^*\mathbb{CP}^1$ as

$$\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=2} = \langle \gamma_{1,1}^l \rangle_{T^*\mathbb{CP}^1} \quad (3.23)$$

with the parameter identification (3.13). Here, we directly compute $\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=2}$ from localization formula (3.8), also $\langle \gamma_{1,1}^l \rangle_{T^*\mathbb{CP}^1}$ by using (3.22), and check their relation (3.23).

$$\int_{T^*\mathbb{CP}^1} [1] = \sum_{j=1}^2 \prod_{\substack{i=1 \\ i \neq j}}^2 \frac{-1}{(z_j - z_i + h)(z_j - z_i)}, \quad (3.24)$$

$$\int_{T^*\mathbb{CP}^1} [\gamma_{1,1}] = \sum_{j=1}^2 \prod_{\substack{i=1 \\ i \neq j}}^2 \frac{-z_j}{(z_j - z_i + h)(z_j - z_i)} \quad (3.25)$$

On the other hand, we evaluated $\langle 1 \rangle_{\text{A-twist}}^{N_c=1, N_f=2}$ in order by order of q , which suggest the q -correction vanish and given by

$$\langle 1 \rangle_{\text{A-twist}}^{N_c=1, N_f=2} = \sum_{j=1}^2 \prod_{\substack{i=1 \\ i \neq j}}^2 \frac{-1}{(z_i - z_j)(z_i - z_j + h)}, \quad (3.26)$$

$$\langle \sigma \rangle_{\text{A-twist}}^{N_c=1, N_f=2} = \sum_{j=1}^2 \prod_{\substack{i=1 \\ i \neq j}}^2 \frac{-z_i}{(z_i - z_j)(z_i - z_j + h)} \quad (3.27)$$

Here we fixed the overall sign by requiring to reproduce the equivariant integration $\int_{T^*\mathbb{CP}^1} [1]$. This shows

$$\langle 1 \rangle_{\text{A-twist}}^{N_c=1, N_f=2} = \langle 1 \rangle_{T^*\mathbb{CP}^1}, \quad (3.28)$$

$$\langle \sigma \rangle_{\text{A-twist}}^{N_c=1, N_f=2} = \langle \gamma_{1,1} \rangle_{T^*\mathbb{CP}^1} \quad (3.29)$$

We computed $\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=2}$, $l = 2, 3, 4, 5$ perturbatively and have agreement of (3.23).

$$\underline{T^*\mathbb{CP}^{n-1}}$$

We conjecture that the expectation value of σ^l agrees with the integration of $\gamma_{1,1}^l \in QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}(T^*\mathbb{CP}^{n-1}; \mathbb{C})$:

$$\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=n} = \langle \gamma_{1,1}^l \rangle_{T^*\mathbb{CP}^{n-1}}, \quad (3.30)$$

$$(3.31)$$

From the ideal, we obtain the following relation

$$\prod_{i=1}^n (\gamma_{1,c} - z_i + h) + \mathbf{q} \prod_{i=1}^n (\gamma_{1,c} - z_i) = 0 \quad (3.32)$$

This relation is same as chiral ring relation via (3.13). From (3.32), $\gamma_{1,1}^l$ with $l > n - 1$ is uniquely expressed as

$$\gamma_{1,1}^l = \sum_{k=0}^{n-1} A_l^{(k)}(z, h, \mathbf{q}) \gamma_{1,1}^k \quad (3.33)$$

With the identification $\sigma = \gamma_{1,1}$, we expect that $\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=n}$ agrees with the equivariant integrals of $\gamma_{1,1}^l$ on quantum cohomology, which means

$$\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=n} = \sum_{k=0}^{n-1} A_l^{(k)}(z, h, \mathbf{q}) \int_{T^*\mathbb{CP}^{n-1}} [\gamma_{1,1}^k] \quad (3.34)$$

We also checked several examples, $n = 3, 4$, also several higher power of σ and found agreement.

3.2.2 $T^*\text{Gr}(r, n)$ with $r \leq n - r$

Here we consider the case $r \leq n - r$. The case $r > n - r$ will be discussed in 3.2.3. We expect that the way of calculation for equivariant integration discussed previous cases can be applied to the non-Abelian cases with $r \leq n - r$. For non-Abelian cases, we use the ideal of quantum cohomology to reduce the degree of symmetric polynomials of $\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}$, which correspond to gauge invariant operators in GLSM side, differently from the abelian case where the Bethe ansatz equation was enough.

As an example, we consider $T^*\text{Gr}(2, 4)$. The generic form of the symmetric polynomial take the form of $(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l$ where k and l are non-negative integers. When $k + l \geq 3$, $(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l$ can be reduced to $(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l$ with $k + l \leq 2$ by using the ideal. This implies that $\langle (\sigma_1 + \sigma_2)^k (\sigma_1\sigma_2)^l \rangle_{\text{A-twist}}^{N_c=2, N_f=4}$ with $k + l \leq 2$ do not have q -dependence and agree with the equivariant integrations of cohomology class $[(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l] \in H_{GL_4(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(2, 4); \mathbb{C})$

$$\langle (\sigma_1 + \sigma_2)^k (\sigma_1\sigma_2)^l \rangle_{\text{A-twist}}^{N_c=2, N_f=4} = \int_{T^*\text{Gr}(2,4)} [(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l] \quad (3.35)$$

On the other hand, when $k + l \geq 3$, the degree of $(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l$ is reduced to

$$(\gamma_{1,1} + \gamma_{1,2})^k (\gamma_{1,1}\gamma_{1,2})^l = \sum_{s+t \leq 2} A_{k,l}^{(s,t)}(z, h, \mathbf{q}) (\gamma_{1,1} + \gamma_{1,2})^s (\gamma_{1,1}\gamma_{1,2})^t \quad (3.36)$$

Therefore, we expect

$$\langle (\sigma_1 + \sigma_2)^k (\sigma_1 \sigma_2)^l \rangle_{\text{A-twist}}^{N_c=2, N_f=4} = \sum_{s+t \leq 2} A_{k,l}^{(s,t)}(z, h, \mathbf{q}) \int_{T^*\text{Gr}(2,4)} [(\gamma_{1,1} + \gamma_{1,2})^s (\gamma_{1,1}\gamma_{1,2})^t] \quad (3.37)$$

We have checked above statements for $k + l \leq 3$ perturbatively. The detail calculation of the reduction (3.36) is available in Appendix. We also checked the cases $k + l = 4$ and some of $k + l = 5$ for $T^*\text{Gr}(2, 5)$ and found agreement.

3.2.3 $T^*\text{Gr}(r, n)$ with $r > n - r$ and Seiberg-like duality

From the ideal $p_1 = 0$ for $T^*\text{Gr}(r, n)$, we have

$$\sum_{a=1}^{\lambda_1} \gamma_{1,a} + \sum_{a=1}^{\lambda_2} \gamma_{2,a} = \frac{(\lambda_1 - \lambda_2) \mathbf{q} h}{(1 - \mathbf{q})} + \sum_{i=1}^n z_i \quad (3.38)$$

In section 3.2.2, we expect, for example,

$$\left\langle \sum_{a=1}^r \sigma_a \right\rangle_{\text{A-twist}}^{N_c=r, N_f=n} = \int_{T^*\text{Gr}(r,n)} \sum_{a=1}^r \gamma_{1,a} \quad \text{for } r \leq n - r, \quad (3.39)$$

i.e. the equivariant integration on quantum cohomology $\langle \sum_{a=1}^r \gamma_{1,a} \rangle_{T^*\text{Gr}(r,n)}$ does not have any q -correction and is simply given by the equivariant integration $\int_{T^*\text{Gr}(r,n)} \sum_{a=1}^r \gamma_{1,a}$. From the relation (3.38), this implies that $\langle \sum_{a=1}^{\lambda_2} \gamma_{2,a} \rangle_{T^*\text{Gr}(r,n)}$ for $r > n - r$ receives q -correction and differs from classical equivariant integration. To clarify the above implication, it is useful to study Seiberg-like duality for twisted chiral ring elements.

Since $T^*\text{Gr}(r, n) \simeq T^*\text{Gr}(n - r, n)$, it is natural to expect that there exist Seiberg-like duality [23] between A-twisted GLSMs with gauge groups $U(r)$ and $U(n - r)$. From (3.6), we find

$$\int_{T^*\text{Gr}(r,n)} [1] = \int_{T^*\text{Gr}(n-r,n)} [1] \quad (3.40)$$

This relation implies

$$\langle 1 \rangle_{\text{A-twist}}^{N_c=r, N_f=n} = \langle 1 \rangle_{\text{A-twist}}^{N_c=n-r, N_f=n} = \int_{T^*\text{Gr}(r,n)} [1] \quad (3.41)$$

We computed each side of (3.41) for $(N_c, N_f) = (1, 3), (1, 4), (2, 5)$ in several orders of q and obtained the agreement.

Next we find the relations between ideals, the equivariant quantum cohomology of $T^*\text{Gr}(n-r, n)$ is given by

$$QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(n-r, n); \mathbb{C}) \simeq \mathbb{C}[\tilde{z}, \tilde{\Gamma}, \tilde{h}]^{S_n \times S_{\tilde{\lambda}_1} \times S_{\tilde{\lambda}_2}} \otimes \mathbb{C}[[\tilde{q}]] / \tilde{\mathcal{I}}_q \quad (3.42)$$

where we use tilde to distinguish the notations from $QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C})$. The ideal $\tilde{\mathcal{I}}^q$ is generated by the n polynomials \tilde{p}_l defined by

$$\begin{aligned} \sum_{l=1}^n \tilde{p}_l(\tilde{z}, \tilde{\Gamma}, \tilde{h}, \tilde{q}) u^{n-l} &:= \prod_{a=1}^2 \prod_{b=1}^{\tilde{\lambda}_a} (u - \tilde{\gamma}_{a,b}) \\ &- \tilde{q} \prod_{a=1}^{\tilde{\lambda}_1} (u - \tilde{\gamma}_{1,a} - h) \prod_{b=1}^{\tilde{\lambda}_2} (u - \tilde{\gamma}_{2,b} + \tilde{h}) - (1 - \tilde{q}) \prod_{i=1}^n (u - \tilde{z}_i) \end{aligned} \quad (3.43)$$

The ideal quantum cohomology of $T^*\text{Gr}(r, n)$ is same as that of $T^*\text{Gr}(n-r, n)$ with the following parameter identification

$$\gamma_{1,a} = \tilde{\gamma}_{2,a} - h, \quad \gamma_{2,a} = \tilde{\gamma}_{1,a} + h, \quad z_i = \tilde{z}_i, \quad h = \tilde{h}, \quad q = \tilde{q}^{-1}. \quad (3.44)$$

This identification was given in [23]. When the equivariant parameters are turned off, $\gamma_{1,a}$ and $\tilde{\gamma}_{2,a}$ are exchanged each other under $T^*\text{Gr}(r, n) \leftrightarrow T^*\text{Gr}(n-r, n)$. This is consistent with the fact that vector bundles with fibers F_1 and F_2/F_1 are exchanged vice versa under $\text{Gr}(r, n) \leftrightarrow \text{Gr}(n-r, n)$.

Next we identify the variables in $QH_{GL_n(\mathbb{C}) \times \mathbb{C}^*}^*(T^*\text{Gr}(r, n); \mathbb{C})$ with $U(n-r)$ GLSM. We substitute $u = \tilde{\gamma}_{1,c}$ and $\tilde{\gamma}_{1,c} + h$ into $\sum_{l=1}^n \tilde{p}_l(\tilde{z}, \tilde{\Gamma}, \tilde{h}, \tilde{q}) u^{n-l} = 0$. Then we obtain

$$\tilde{q} \prod_{\substack{a=1 \\ a \neq c}}^{\tilde{\lambda}_1} \frac{\tilde{\gamma}_{1,c} - \tilde{\gamma}_{1,a} + \tilde{h}}{\tilde{\gamma}_{1,c} - \tilde{\gamma}_{1,a} - h} = \prod_{i=1}^n \frac{\tilde{\gamma}_{1,c} - \tilde{z}_i}{\tilde{\gamma}_{1,c} - \tilde{z}_i + \tilde{h}} \quad (3.45)$$

With the identification

$$n-r = N_c, \quad n = N_f, \quad \tilde{\gamma}_{1,a} = \tilde{\sigma}_a, \quad z_i = \tilde{m}_i + \frac{\tilde{m}^z}{2}, \quad \tilde{h} = \tilde{m}^z, \quad \tilde{q} = (-1)^{N_f} \tilde{q}. \quad (3.46)$$

(3.45) also agrees with the twisted chiral ring relation.⁸ Using tilde to the parameters in $U(n-r)$ GLSM, (3.13) and (3.46) provide the relation of twisted mass parameters and FI-parameter in $U(r)$ and $U(n-r)$ GLSM

$$m_i = \tilde{m}_i, \quad m^z = \tilde{m}^z, \quad q = \tilde{q}^{-1}. \quad (3.47)$$

With the identification (3.44), we have [23]

$$\begin{aligned} \sum_{a=1}^r \gamma_{1,a} &= \sum_{a=1}^{n-r} \gamma_{2,a} + \frac{(2r-n)qh}{(1-q)} + \sum_{i=1}^n z_i \\ &= \sum_{a=1}^{n-r} (\tilde{\gamma}_{1,a} + h) + \frac{(2r-n)qh}{(1-q)} + \sum_{i=1}^n z_i \end{aligned} \quad (3.48)$$

⁸There is another way of identification, but considering Seiberg-like duality above identification is more appropriate.

This gives the map between the σ in $U(r)$ and $U(n-r)$ GLSMs.

$$\sum_{a=1}^r \sigma_a = \sum_{a=1}^{n-r} (\tilde{\sigma}_a + h) + \frac{(2r-n)\mathbf{q}h}{(1-\mathbf{q})} + \sum_{i=1}^n z_i \quad (3.49)$$

Therefore we obtain

$$\left\langle \sum_{a=1}^r \sigma_a \right\rangle_{\text{A-twist}}^{N_c=r, N_f=n} = \left\langle \sum_{a=1}^{n-r} (\tilde{\sigma}_a + h) + \frac{(2r-n)\mathbf{q}h}{(1-\mathbf{q})} + \sum_{i=1}^n z_i \right\rangle_{\text{A-twist}}^{N_c=n-r, N_f=n} \quad (3.50)$$

We evaluate the left hand side of (3.50) in $|q| < 1$. From $q = \tilde{q}^{-1}$, the right hand side is evaluated in the region $|\tilde{q}| > 1$ in which the one-loop determinant of anti-chiral multiplets contribute to Jeffrey–Kirwan residues. Note that $\langle \sum_{a=1}^{n-r} (\tilde{\sigma}_a + h) \rangle_{\text{A-twist}}^{N_c=n-r, N_f=n}$ with $|\tilde{q}| > 1$ is same as $\langle \sum_{a=1}^{n-r} \tilde{\sigma}_a \rangle_{\text{A-twist}}^{N_c=n-r, N_f=n}$ with $|\tilde{q}| < 1$ by parameter change $\tilde{q}^{-1} \rightarrow \tilde{q}$, but from (3.39), $\langle \sum_{a=1}^{n-r} \tilde{\sigma}_a \rangle_{\text{A-twist}}^{N_c=n-r, N_f=n}$ for $r > n-r$ does not have \tilde{q} -correction. Thus we have

$$\left\langle \sum_{a=1}^{n-r} (\tilde{\sigma}_a + h) + \frac{(2r-n)\mathbf{q}h}{(1-\mathbf{q})} + \sum_{i=1}^n z_i \right\rangle_{\text{A-twist}}^{N_c=n-r, N_f=n} \quad (3.51)$$

$$= \sum_{I_{n-r} \subset I} \prod_{a \in I_{n-r}} \prod_{b \in I_r} \frac{(-1)^{r(n-r)}}{(z_a - z_b)(z_a - z_b + h)} \left(\sum_{c \in I_r} z_c + \frac{(2r-n)\mathbf{q}h}{(1-\mathbf{q})} \right), \quad (3.52)$$

where $r > n-r$ and $|\tilde{q}| > 1$. On the other hand, the correspondence between A-twisted GLSM and quantum cohomology gives

$$\left\langle \sum_{a=1}^r \gamma_{1,a} \right\rangle_{T^*\text{Gr}(r,n)} = \left\langle \sum_{a=1}^{N_c} \sigma_a \right\rangle_{\text{A-twist}}^{N_c=r, N_f=n} \quad (3.53)$$

Therefore from (3.48) and (3.53) we obtain

$$\left\langle \sum_{a=1}^r \gamma_{1,a} \right\rangle_{T^*\text{Gr}(r,n)} = \left\langle \sum_{a=1}^{N_c} \sigma_a \right\rangle_{\text{A-twist}}^{N_c=r, N_f=n} \quad (3.54)$$

$$= \sum_{I_{n-r} \subset I} \prod_{a \in I_{n-r}} \prod_{b \in I_r} \frac{(-1)^{r(n-r)}}{(z_a - z_b)(z_a - z_b + h)} \left(\sum_{c \in I_r} z_c + \frac{(2r-n)\mathbf{q}h}{(1-\mathbf{q})} \right) \quad (3.55)$$

We computed $\left\langle \sum_{a=1}^r \sigma_a \right\rangle_{\text{A-twist}}^{N_c=r, N_f=n}$ in $|q| < 1$ for $(N_c, N_f) = (2, 3), (2, 4), (3, 4)$ in several order of q and confirm the equality.

Next we consider $\text{Tr}_{A_2} \sigma$. Here trace is taken over the second anti-symmetric representation. To make the expression compact, we call $e_l(\gamma_1), e_l(\tilde{\gamma}_1), e_l(\gamma_2), e_l(\tilde{\gamma}_2), e_l(\sigma), e_l(\tilde{\sigma})$

and $e_l(z)$ as the l -th elementary symmetric polynomials of $\gamma_{1,a}, \tilde{\gamma}_{1,a}, \gamma_{2,a}, \tilde{\gamma}_{1,a}, \sigma_a, \tilde{\sigma}_a$ and z_i , respectively. We can eliminate $e_1(\gamma_2)$ from $p_2 = 0$ by using $p_1 = 0$. Then we obtain

$$\begin{aligned}
(1 - \mathbf{q})[e_2(\gamma_2) + e_2(\gamma_1) - (e_1(\gamma_2))^2 + e_1(z)e_1(\gamma_2) - e_2(z)] \\
= -(\lambda_1 - \lambda_2 - 2)\mathbf{q}he_1(\gamma_2) + (\lambda_1 - \lambda_2 - 1)\mathbf{q}he_1(z) \\
+ \frac{\mathbf{q}h^2}{2}(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2)\mathbf{q}^2h^2}{1 - \mathbf{q}}
\end{aligned} \tag{3.56}$$

From (3.44), this give rise to Seiberg-like duality for $\text{Tr}_{A_2}\sigma$:

$$\begin{aligned}
(1 - \mathbf{q})\langle e_2(\sigma) \rangle_{\text{A-twist}}^{N_c=r, N_f=n} \\
= \left\langle (\mathbf{q} - 1)[e_2(\tilde{\sigma} + h) - (e_1(\tilde{\sigma} + h))^2 + e_1(z)e_1(\tilde{\sigma} + h) - e_2(z)] - (\lambda_1 - \lambda_2 - 2)\mathbf{q}he_1(\tilde{\sigma} + h) \right. \\
\left. + (\lambda_1 - \lambda_2 - 1)\mathbf{q}he_1(z) + \frac{\mathbf{q}h^2}{2}(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2)\mathbf{q}^2h^2}{1 - \mathbf{q}} \right\rangle_{\text{A-twist}}^{N_c=n-r, N_f=n}
\end{aligned} \tag{3.57}$$

Moreover, one can determine the explicit form of the left hand side for $r > n - r$ in terms of Seiberg-like duality. We evaluate the left hand side in the region $|q| < 1$, $\langle e_l(\tilde{\sigma} + h) \rangle_{\text{A-twist}}^{N_c=n-r, N_f=n}$ in the right hand side does not have \tilde{q} -correction and is written as

$$\begin{aligned}
\langle e_2(\sigma) \rangle_{\text{A-twist}}^{N_c=r, N_f=n} &= \sum_{I_{n-r} \subset I} \prod_{a \in I_{n-r}} \prod_{b \in I_r} \frac{(-1)^{r(n-r)}(1 - \mathbf{q})^{-1}}{(z_a - z_b)(z_a - z_b + h)} \\
&\times \left((\mathbf{q} - 1)[e_2(\mathbf{z}) - (e_1(\mathbf{z}))^2 + e_1(z)e_1(\mathbf{z}) - e_2(z)] \right. \\
&\quad - (\lambda_1 - \lambda_2 - 2)\mathbf{q}he_1(\mathbf{z}) + (\lambda_1 - \lambda_2 - 1)\mathbf{q}he_1(z) \\
&\quad \left. - \frac{\mathbf{q}h^2}{2}(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2)\mathbf{q}^2h^2}{1 - \mathbf{q}} \right)
\end{aligned} \tag{3.58}$$

Here $e_l(\mathbf{z})$ means $e_l(z_{i_1}, \dots, z_{i_{n-r}})$ with $I_{n-r} = \{i_1, \dots, i_{n-r}\}$. For examples, in $(\lambda_1, \lambda_2) = (r, n - r) = (2, 3)$ case

$$\begin{aligned}
\langle e_2(\sigma) \rangle_{\text{A-twist}}^{N_c=2, N_f=3} &= \sum_{I_1 \subset I} \prod_{a \in I_1} \prod_{b \in I_r} \frac{1}{(z_a - z_b)(z_a - z_b + h)} \\
&\times \left((e_1(\mathbf{z}))^2 - e_1(z)e_1(\mathbf{z}) + e_2(z) + \frac{\mathbf{q}h}{1 - \mathbf{q}}e_1(\mathbf{z}) - \frac{\mathbf{q}h^2}{1 - \mathbf{q}} \right),
\end{aligned} \tag{3.59}$$

In the similar manner, we can eliminate $e_l(\gamma_1), (l = 1, \dots, k - 1)$ from $p_k = 0$ by using $p_l = 0, (l = 1, \dots, k - 1)$ and recursively derive Seiberg-like duality between $e_k(\sigma_1)$ and $e_l(\tilde{\sigma}), (l = 1, \dots, k)$.

4 Wilson loops in 3d $\mathcal{N} = 2^*$ theory and Bethe subalgebra of XXZ model

In previous section, we saw that the twisted chiral ring relation of GLSM is given by equivariant quantum cohomology ring, which corresponds to Bethe subalgebra of XXX spin

chain model, by calculating correlation functions of gauge invariant operators. Therefore we can do similar calculations and checks for the chiral ring relation of 3d $\mathcal{N} = 2^*$ theory and Bethe subalgebra of XXZ spin chain model [10]. In 3d $\mathcal{N} = 2^*$ theory on $S^1 \times S^2$, chiral ring is generated by Wilson loops wrapped on S^1 .

In 3d $\mathcal{N} = 2^*$ theory, which is obtained by adjoint mass deformation of 3d $\mathcal{N} = 4$ theory, there is a superpotential

$$W_{\tilde{Q}\Phi Q} = \sum_{a,b=1}^{N_c} \sum_{i=1}^{N_f} \tilde{Q}_i^a \Phi_a^b Q_b^i. \quad (4.1)$$

This breaks $SU(N_f)_Q \times SU(N_f)_{\tilde{Q}} \times U(1)_D$ to $SU(N_f) \times U(1)_D$. Also, we turn off all the background magnetic fluxes for flavor symmetries. We denote the fugacity for $SU(N_f)$ flavor symmetry by y_i and the one for $U(1)_D$ flavor symmetry by z .

	$U(N_c)$	$SU(N_f)$	$U(1)_D$	$U(1)_T$	$U(1)_R$
Q	N_c	\overline{N}_f	$-1/2$	0	0
\tilde{Q}	\overline{N}_c	N_f	$-1/2$	0	0
Φ	adj	1	1	0	2

Table 4. Matter contents of 3d $\mathcal{N} = 2$ theory

The expectation values of supersymmetric Wilson loop in the representation \mathcal{R} is given by

$$\begin{aligned} \langle W_{\mathcal{R}} \rangle_{\text{top. twisted}}^{N_c, N_f} &= \frac{1}{N_c!} \sum_{\vec{m} \in \mathbb{Z}^{N_c}} \int \prod_{a=1}^{N_c} \frac{dx_a}{2\pi i x_a} \text{Tr}_{\mathcal{R}}(x) \prod_{a \neq b} \left(1 - \frac{x_a}{x_b}\right) \prod_{a,b} \left(\frac{x_a^{1/2} x_b^{-1/2} z^{1/2}}{1 - x_a x_b^{-1} z} \right)^{m_a - m_b - 1} \\ &\quad \times \prod_{a=1}^{N_c} \zeta^{m_a} \prod_{i=1}^{N_f} \left(\frac{x_a^{1/2} y_i^{-1/2} z^{-1/4}}{1 - x_a y_i^{-1} z^{-1/2}} \right)^{m_a + 1} \left(\frac{x_a^{-1/2} y_i^{1/2} z^{-1/4}}{1 - x_a^{-1} y_i z^{-1/2}} \right)^{-m_a + 1} \end{aligned} \quad (4.2)$$

Here we absorbed $(-1)^{N_c-1}$ into the definition of ζ . When the representation \mathcal{R} is the l -th anti-symmetric representation A_l , $\text{Tr}_{A_l}(x)$ is given by the l -th elementary symmetric polynomial of $(x) = \text{diag}(x_1, \dots, x_{N_c})$

$$\text{Tr}_{A_k}(x) = \sum_{1 \leq a_1 < \dots < a_l \leq N_c} x_{a_1} \cdots x_{a_l} \quad (4.3)$$

Note that any product of supersymmetric Wilson loops is a symmetric function of (x) , which is also expressed in terms of the elementary symmetric polynomials.

4.1 Bethe subalgebra of XXZ spin chain model

It was shown in [10] that Bethe subalgebra of XXZ model is given by the algebra $\mathcal{K}^{\mathfrak{q}}$,

$$\mathcal{K}^{\mathfrak{q}} = \mathbb{C}[z^{\pm}, \Gamma^{\pm}, h^{\pm}]^{S_{\lambda_1} \times S_{\lambda_2}} \otimes \mathbb{C}[[\mathfrak{q}]] / \mathcal{I}_{\mathfrak{q}} \quad (4.4)$$

where $\mathbb{C}[\mathbf{z}^\pm, \Gamma^\pm, \mathbf{h}^\pm]$ is Laurent polynomial ring of $\mathbf{z}^\pm := \{z_1^\pm, \dots, z_n^\pm\}$ and $\Gamma^\pm = (\Gamma_1^\pm, \Gamma_2^\pm)$ with $\Gamma_1^\pm := \{\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}\}$ and $\Gamma_2^\pm := \{\gamma_{2,1}, \dots, \gamma_{2,\lambda_2}\}$. Here we take $(\lambda_1, \lambda_2) = (r, n)$. The S_{λ_i} in the exponent denotes symmetrization of the variable Γ_i^\pm . Also, the ideal $\mathcal{I}_{\mathbf{q}}$ is generated by the n coefficients of the following polynomial P of u^{-1}

$$P(\Gamma, \mathbf{z}, \mathbf{h}, \mathbf{q}) := (1 - \mathbf{q}) \prod_{i=1}^n (1 - u^{-1} \mathbf{z}_i) - \prod_{i=1}^2 \prod_{a=1}^{\lambda_i} (1 - u^{-1} \gamma_{i,a}) + \mathbf{q} \prod_{a=1}^{\lambda_1} (1 - u^{-1} \mathbf{h}^{-1} \gamma_{1,a}) \prod_{b=1}^{\lambda_2} (1 - u^{-1} \mathbf{h} \gamma_{2,b}). \quad (4.5)$$

So far we do not know the rigorous geometric interpretation of (4.4), but it was conjectured in [10] that the algebra $\mathcal{K}^{\mathbf{q}}$ is isomorphic to the equivariant quantum K -theory algebra of the cotangent bundle of Grassmannian $QK_{GL_n(\mathbb{C}^*) \times \mathbb{C}^*}(T^* \text{Gr}(r, n), \mathbb{C})$.

Bethe ansatz equation and match of parameters

We consider how the generators of $\mathcal{K}^{\mathbf{q}}$ are identified with the variables in topologically twisted 3d $\mathcal{N} = 2^*$ supersymmetric theory by deriving the Bethe ansatz equation from (4.5).

By substituting $u = \gamma_{1,a}$ and $\gamma_{1,a} \mathbf{h}^{-1}$ into $P(\Gamma, \mathbf{z}, \mathbf{h}, \mathbf{q}) = 0$, we obtain, respectively,

$$(1 - \mathbf{q}) \prod_{i=1}^n (1 - \gamma_{1,a}^{-1} \mathbf{z}_i) = -\mathbf{q} \prod_{b=1}^{\lambda_1} (1 - \mathbf{h}^{-1} \gamma_{1,a}^{-1} \gamma_{1,b}) \prod_{b=1}^{\lambda_2} (1 - \mathbf{h} \gamma_{1,a}^{-1} \gamma_{2,b}) \quad (4.6)$$

$$(1 - \mathbf{q}) \prod_{i=1}^n (1 - \mathbf{h} \gamma_{1,a}^{-1} \mathbf{z}_i) = \prod_{i=1}^2 \prod_{b=1}^{\lambda_i} (1 - \mathbf{h} \gamma_{1,a}^{-1} \gamma_{i,b}) \quad (4.7)$$

Dividing (4.6) by (4.7), we get

$$\prod_{i=1}^n \frac{\gamma_{1,a} - \mathbf{z}_i}{\gamma_{1,a} - \mathbf{h} \mathbf{z}_i} = \mathbf{h}^{-1} \mathbf{q} \prod_{\substack{b=1 \\ b \neq a}}^{\lambda_1} \frac{\gamma_{1,a} - \mathbf{h}^{-1} \gamma_{1,b}}{\gamma_{1,a} - \mathbf{h} \gamma_{1,b}} \quad (4.8)$$

We find that (4.8) is same as the relation $\zeta e^{iB_a} = 1$ of the supersymmetric Wilson loop with the following identification

$$r = N_c, \quad n = N_f, \quad \gamma_{1,a} = x_a, \quad \mathbf{z}_i = y_i z^{\frac{1}{2}}, \quad \mathbf{h} = z^{-1}, \quad \mathbf{q} = z^{\frac{N_f}{2} - N_c} \zeta. \quad (4.9)$$

4.2 Properties of Wilson loop expectation values in topologically twisted 3d $\mathcal{N} = 2^*$ theory

Here, we provide the explicit calculation of Wilson loop expectation value (4.2) for several examples and see that they indeed satisfy (4.4).

Abelian cases

From (4.8), which is equivalent to $\zeta e^{iB_a} = 1$, we expect that the supersymmetric Wilson loop $W = x$ for $U(1)$ gauge theories satisfy the following relation

$$\left\langle \prod_{i=1}^n (W - z_i) - qh^{-1}(W - hz_i) \right\rangle^{N_c=1, N_f=n} = 0 \quad (4.10)$$

Here we used the parameter identification (4.9). By using (4.8), the higher order correlation functions $\langle W^l \rangle^{N_c=1, N_f}$ for $l \geq n$ is expressed in terms of W^k , ($k = 0, 1, \dots, n-1$) as

$$\langle W^l \rangle^{N_c=1, N_f=n} = \sum_{k=0}^{l-1} A_l^{(k)}(z, h, q) \langle W^k \rangle^{N_c=1, N_f=n} \quad (4.11)$$

In the 2d $\mathcal{N} = (2, 2)^*$ theory with $N_f = n$ flavors, we found $\langle \sigma^l \rangle_{\text{A-twist}}^{N_c=1, N_f=n}$, ($l \leq n-1$) do not have q -correction. We have a similar property in 3d $\mathcal{N} = 2^*$ theory. For $0 \leq l \leq n-1$, $\langle W^l \rangle^{N_c=1, N_f=n}$ does not have ζ -correction and is given by the zero magnetic charge sector as

$$\langle W^l \rangle^{N_c=1, N_f=n} = \sum_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(-1)^{n-1} z_i^l}{\left((z_i z_j^{-1})^{\frac{1}{2}} - (z_i z_j^{-1})^{-\frac{1}{2}} \right) \left((h^{-1} z_i z_j^{-1})^{\frac{1}{2}} - (h^{-1} z_i z_j^{-1})^{-\frac{1}{2}} \right)} \quad (4.12)$$

We have check (4.10) and (4.12) for $N_f = 2, 3, 4$ in several orders of ζ .

Non-Abelian cases

In two dimension, we observed that the partition function does not receive any q -correction and is given by the residue at the zero magnetic charge sector. Similarly we observed that the partition function (index) of 3d $\mathcal{N} = 2^*$ theory does not receive ζ -correction neither and is given by

$$\langle 1 \rangle^{N_c=r, N_f=n} = \sum_{I_r \subset I} \prod_{i \in I_r} \prod_{j \in I_{n-r}} \frac{(-1)^{(n-r)r}}{\left((z_i z_j^{-1})^{\frac{1}{2}} - (z_i z_j^{-1})^{-\frac{1}{2}} \right) \left((h^{-1} z_i z_j^{-1})^{\frac{1}{2}} - (h^{-1} z_i z_j^{-1})^{-\frac{1}{2}} \right)} \quad (4.13)$$

In two dimensions, $\langle 1 \rangle_{\text{A-twist}}^{N_c=r, N_f=n}$ has a geometrical interpretation as equivariant integration of $[1] \in H^*(T^*\text{Gr}(r, n); \mathbb{C})$. (4.13) also have geometrical interpretation as follows. If we identify 3d parameter z_i and h as $z_i = e^{z_i}$ and $h = e^{-h}$, (4.13) is sinh uplift of the equivariant integration which can be interpreted as equivariant Dirac index.

For $2 < n-2$, we also observed that the expectation values of $x_1^{\pm 1} + x_2^{\pm 1}$, $(x_1 x_2^{-1})^{\pm 1} + (x_1^{-1} x_2)^{\pm 1}$ and $(x_1 x_2)^{\pm 1}$ do not have ζ -correction. For example, the expectation value of $W_F^{\pm 1} = x_1^{\pm 1} + x_2^{\pm 1}$ is given as

$$\langle W_F^{\pm 1} \rangle^{N_c=2, N_f=n} = \sum_{I_2 \subset I} \prod_{i \in I_2} \prod_{j \in I_{n-2}} \frac{(\sum_{a \in I_2} z_a^{\pm 1})}{\left((z_i z_j^{-1})^{\frac{1}{2}} - (z_i z_j^{-1})^{-\frac{1}{2}} \right) \left((h^{-1} z_i z_j^{-1})^{\frac{1}{2}} - (h^{-1} z_i z_j^{-1})^{-\frac{1}{2}} \right)} \quad (4.14)$$

On the other hand, the properties of the correlation function $(x_1 + x_2)^2$, $(x_1 + x_2)(x_1 x_2)$, and $(x_1 x_2)^2$ are different from the two dimensions. In the 2d $\mathcal{N} = (2, 2)^*$ theory with $N_c < N_f - N_c$, we expected that the correlation functions of symmetric polynomial of σ

$$\langle \prod_{a=1}^{N_c} e_a^{l_a}(\sigma) \rangle \text{ with } \sum_{a=1}^{N_c} l_a \leq N_c, \quad (4.15)$$

don't have q -dependence. This may be interpreted as that the degree of polynomial is not reducible in the polynomial ring by the ideal. For example, $\langle (\sigma_1 + \sigma_2)^l (\sigma_1 \sigma_2)^k \rangle^{N_c=2, N_f}$ with $k+l=2$ agrees the residues at zero magnetic charges sector. do not have q -dependence. For examples $\langle (\sigma_1 + \sigma_2)^l (\sigma_1 \sigma_2)^k \rangle^{N_c=2, N_f=4}$ with $k+l=2$ agrees the residues at zero magnetic charges sector. On the other hand, if we eliminate $\gamma_{2,1} + \gamma_{2,2}$ and $\gamma_{2,1}\gamma_{2,2}$ from the ideal of \mathcal{K}^q , we obtain the following relation

$$\frac{(1 - h^2 q) e_4(z)}{e_2(\gamma_1)} + e_2(\gamma_1) \left(1 - \frac{q}{h^2}\right) + \frac{(q-1)[e_1^2(\gamma_1)(q-h) - h(q-1)e_1(\gamma_1)e_1(z)]}{h(hq-1)} + (q-1)e_2(z) = 0 \quad (4.16)$$

$$\frac{(hq-1)e_4(z)e_1(\gamma_1)}{e_2(\gamma_1)} - (h-q)e_2(\gamma_1) \frac{(h-q)e_1(\gamma_1) + h(q-1)e_1(z)}{h^2(hq-1)} + (1-q)e_3(z) = 0 \quad (4.17)$$

Then we find that the degree of $\langle (x_1 + x_2)^l (x_1 x_2)^k \rangle^{N_c=2, N_f=4}$ with $k+l=2$ is reduced by the above equation and $(x_1 + x_2)^l (x_1 x_2)^k$ has ζ dependence.

4.3 Wilson loops and Seiberg-like duality in 3d $\mathcal{N} = 2^*$ theory

From the expression of (4.13), the partition functions (index) agree in the $U(N_c)$ and $U(N_f - N_c)$ gauge theory

$$\langle 1 \rangle^{N_c, N_f} = \langle 1 \rangle^{N_f - N_c, N_f} \quad (4.18)$$

To find the map of Wilson loops between $U(r)$ and $U(n-r)$ gauge theory, let us consider polynomial $P(\tilde{\Gamma}, \tilde{z}, \tilde{h}, \tilde{q}) = 0$ for $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (n-r, n)$

$$(1 - \tilde{q}) \prod_{i=1}^n (1 - u^{-1} \tilde{z}_i) = \prod_{i=1}^2 \prod_{a=1}^{\tilde{\lambda}_i} (1 - u^{-1} \tilde{\gamma}_{i,a}) - q \prod_{a=1}^{\tilde{\lambda}_1} (1 - u^{-1} \tilde{h}^{-1} \tilde{\gamma}_{1,a}) \prod_{b=1}^{\tilde{\lambda}_2} (1 - u^{-1} \tilde{h} \tilde{\gamma}_{2,b}). \quad (4.19)$$

In the following identification

$$\lambda_1 = \tilde{\lambda}_2, \quad \lambda_2 = \tilde{\lambda}_1, \quad \gamma_{1,a} = \tilde{\gamma}_{2,a} \tilde{h}, \quad \gamma_{2,a} = \tilde{\gamma}_{1,a} \tilde{h}^{-1}, \quad z_i = \tilde{z}_i, \quad h = \tilde{h}, \quad q = \tilde{q}^{-1}, \quad (4.20)$$

(4.19) is identical to $P(\Gamma, \mathbf{z}, h, q)$. (4.20) give isomorphism between Bethe subalgebras with $(\lambda_1, \lambda_2) = (r, n-r)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (n-r, n)$. Substitution $u = \tilde{\gamma}_{1,a} \tilde{\gamma}_{1,a} \tilde{h}^{-1}$ into gives the relation

$$\prod_{i=1}^n \frac{\tilde{\gamma}_{1,a} - \tilde{z}_i}{\tilde{\gamma}_{1,a} - \tilde{h} \tilde{z}_i} = \tilde{h}^{-1} \tilde{q} \prod_{\substack{b=1 \\ b \neq a}}^{\tilde{\lambda}_1} \frac{\tilde{\gamma}_{1,a} - \tilde{h}^{-1} \tilde{\gamma}_{1,b}}{\tilde{\gamma}_{1,a} - \tilde{h} \tilde{\gamma}_{1,b}}, \quad (4.21)$$

which is again same as $\tilde{\zeta}e^{iB_a} = 1$ with the following identification

$$n - r = N_c, \quad n = N_f, \quad \tilde{\gamma}_{1,a} = \tilde{x}_a, \quad \tilde{z}_i = \tilde{y}_i \tilde{z}^{\frac{1}{2}}, \quad \tilde{h} = \tilde{z}^{-1}, \quad \tilde{q} = \tilde{z}^{\frac{N_f}{2} - N_c} \tilde{\zeta}. \quad (4.22)$$

Here we take tilde for the variables in $U(n - r)$ gauge theory to distinguish from that of $U(r)$ gauge theory. From (4.9), (4.20) and (4.22), we have the maps of the variables between $U(r)$ and $U(n - r)$ 3d $\mathcal{N} = 2^*$ gauge theories:

$$y_i = \tilde{y}_i, \quad z = \tilde{z}, \quad \zeta = \tilde{\zeta}^{-1}. \quad (4.23)$$

From now on, we do not distinguish y_i , z and z_i from \tilde{y}_i , \tilde{z} and \tilde{z}_i . We write down the Seiberg-like duality for supersymmetric Wilson loops. The coefficient of u^{-n+1} of $P = 0$ gives

$$(1 - hq) \sum_{a=1}^{\lambda_2} \gamma_{2,a} = -(1 - h^{-1}q) \left(\sum_{a=1}^{\lambda_1} \gamma_{1,a} \right) - (1 - q) \left(\sum_{i=1}^n z_i \right) \quad (4.24)$$

By using the relations between Bethe subalgebra and fugacities in 3d index, (4.24) leads to Seiberg-like duality for Wilson loop in fundamental representation $W_F = \sum_{a=1}^{N_c} x_a$.

$$(1 - z^{\frac{n}{2} - r - 1} \zeta) z \left(\sum_{a=1}^{n-r} \tilde{x}_a \right) = -(1 - z^{\frac{n}{2} - r + 1} \zeta) \left(\sum_{a=1}^r x_a \right) - (1 - z^{\frac{n}{2} - r} \zeta) \left(\sum_{i=1}^n z_i \right) \quad (4.25)$$

and the Seiberg-like duality for expectation values is given by

$$\begin{aligned} (1 - z^{\frac{n}{2} - r - 1} \zeta) z \langle W_F \rangle^{N_c = n - r, N_f = n} \\ = -(1 - z^{\frac{n}{2} - r + 1} \zeta) \langle W_F \rangle^{N_c = r, N_f = n} - (1 - z^{\frac{n}{2} - r} \zeta) \left(\sum_{i=1}^n z_i \right) \langle 1 \rangle^{N_c = r, N_f = n} \end{aligned} \quad (4.26)$$

For example, the evaluation of the right hand side in the region $\zeta < 1, (\zeta > 1)$ means that the left hand side is evaluated in the region $\tilde{\zeta} = \zeta^{-1} > 1, (\tilde{\zeta} = \zeta^{-1} < 1)$ in which negative (positive) magnetic charge contribute to the Jeffrey-Kirwan residue operations, vice versa. We evaluated the left and right hand side independently and have agreement in several (r, n) .

Next we derive the Seiberg-like duality for second anti-symmetric representation. The derivation is parallel to that in two dimensions. We eliminate $e_1(\gamma_1)$ from the coefficient of u^{-2} in $P = 0$. Then we obtain the relation

$$(1 - h^{-2}q) e_2(\gamma_1) + (1 - h^2q) e_2(\gamma_2) = (1 - q) \left[\frac{(1 - hq)}{(1 - h^{-1}q)} (e_1(\gamma_2))^2 + \frac{(1 - q)}{(1 - h^{-1}q)} e_1(z) e_1(\gamma_2) + e_2(z) \right] \quad (4.27)$$

This can be written in terms of x_a and \tilde{x}_a as

$$\begin{aligned} (1 - h^{-2}q) e_2(x) &= -(1 - h^2q) e_2(\tilde{x} h^{-1}) \\ &+ (1 - q) \left[\frac{(1 - hq)}{(1 - h^{-1}q)} (e_1(\tilde{x} h^{-1}))^2 + \frac{(1 - q)}{(1 - h^{-1}q)} e_1(z) e_1(\tilde{x} h^{-1}) + e_2(z) \right] \end{aligned} \quad (4.28)$$

Therefore we obtain the Seiberg-like duality for the second anti-symmetric representation $W_{A_2} = \sum_{a < b} x_a x_b$ as

$$\begin{aligned} & (1 - h^{-2}q) \langle W_{A_2} \rangle^{N_c=r, N_f=n} \\ &= \left\langle (q - h^2)W_{A_2} + (1 - q) \left[\frac{(h^{-1} - q)}{(h - q)} (W_F)^2 + \frac{(1 - q)}{(h - q)} e_1(z)W_F + e_2(z) \right] \right\rangle^{N_c=n-r, N_f=n} \end{aligned} \quad (4.29)$$

In the similar way, we can obtain Seiberg duality for Wilson loops in other representations.

5 Conclusion and Future Directions

In this paper, we discussed the relation between the partition function in A-twisted 2d $\mathcal{N} = (2, 2)$ and topologically twisted 3d $\mathcal{N} = 2$ gauge theories and the inverse of the norm of Bethe eigenstates for $XXX_{1/2}$ and $XXZ_{1/2}$ spin chain model with a particular choice of R -charges and background magnetic fluxes for flavor symmetries in gauge theory sides. The correlation functions in gauge theories could be understood as the coefficients of the expectation values of Baxter Q -operators as well as conserved charges with appropriate coefficient.

We also studied the relation between correlation functions in A-twisted 2d $\mathcal{N} = (2, 2)^*$ gauge theories and equivariant integration of equivariant quantum cohomology for the cotangent bundle of Grassmannians. A method of calculating the equivariant integration was described in Section 3, which we expect to hold. We also considered Seiberg-like duality of 2d $\mathcal{N} = (2, 2)^*$ to calculate the equivariant integration. We have checked several examples and found that the equivariant integration via the method described above are matched with the correlation functions.

From the 2d $\mathcal{N} = (2, 2)^*$ calculations where chiral ring relation is identified with Bethe subalgebra of XXX spin chain model, we did similar calculation for 3d $\mathcal{N} = 2^*$. We calculated correlation functions of Wilson loops and checked that they agree with Bethe subalgebra of XXZ spin chain model.

There are several interesting directions. Firstly, it will be interesting to find the analogue of the equivariant integration in equivariant quantum K -theory and match them with the correlation functions of Wilson loops in topologically twisted 3d $\mathcal{N} = 2^*$ theory.

Another interesting direction is to study relations between Bethe ansatz and finite dimensional commutative Frobenius algebra. In [24], a finite dimensional commutative Frobenius algebra was constructed in terms of Bethe ansatz for q -boson model. It is known that finite dimensional commutative Frobenius algebra is essentially same as 2d topological quantum field theory (TQFT) and 2d partition functions on genus g Riemann surface Σ_g corresponding to q -boson can be written as [16]

$$Z(\Sigma_g) = \sum_{(\lambda) \in P_{q\text{-boson}}} \langle \Psi(\lambda) | \Psi(\lambda) \rangle^{g-1} \quad (5.1)$$

Here $|\Psi(\lambda)\rangle$ is an eigen vector of q -boson determined by Bethe root (λ) . We obtained same type formulas for XXX and also XXZ spin chains and corresponding TQFT are

topologically twisted 2d $\mathcal{N} = (2, 2)$ and 3d $\mathcal{N} = 2$ theories. By using a recent result [25], the partition function of 2d $\mathcal{N} = (2, 2)$ and 3d $\mathcal{N} = 2$ theories studied in this paper can be generalized to the Riemann surfaces as

$$Z(\Sigma_g) = \sum_{(\lambda) \in P_{\text{XXX}}} \langle \Psi(\lambda) | \Psi(\lambda) \rangle^{g-1} \quad (5.2)$$

$$Z(S^1 \times \Sigma_g) = \sum_{(\lambda) \in P_{\text{XXZ}}} \langle \Psi(\lambda) | \Psi(\lambda) \rangle^{g-1} \quad (5.3)$$

These formulas are very similarly to q -boson case and implies that there exist dimensional commutative Frobenius algebras associated with Bethe ansatz for XXX and also XXZ spin chains. It would be interesting to construct Frobenius algebras in terms of XXX and XXZ spin chains.

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A Reducing the Polynomial Ring by Ideal in $T^*\text{Gr}(2, 4)$

In this appendix, we provide some detail calculation to get (3.36). From the ideal (3.5),

$$(u - \gamma_{1,1})(u - \gamma_{1,2})(u - \gamma_{2,1})(u - \gamma_{2,2}) - \mathbf{q}(\gamma_{1,1} + h - u)(\gamma_{1,2} + h - u)(-\gamma_{2,1} + h + u)(-\gamma_{2,2} + h + u) - (1 - \mathbf{q})(u - z_1)(u - z_2)(u - z_3)(u - z_4) \quad (\text{A.1})$$

we obtain four relations $p_l = 0$, $l = 1, 2, 3, 4$ with

$$p_1 = (\mathbf{q} - 1)(\gamma_{1,1} + \gamma_{1,2} + \gamma_{2,1} + \gamma_{2,2} - z_1 - z_2 - z_3 - z_4) \quad (\text{A.2})$$

$$p_2 = (\gamma_{2,1} + \gamma_{2,2})(\gamma_{1,1} + \gamma_{1,2}) + \gamma_{1,1}\gamma_{1,2} + \gamma_{2,1}\gamma_{2,2} \quad (\text{A.3})$$

$$+ \mathbf{q}(-(\gamma_{2,1} + \gamma_{2,2})(\gamma_{1,1} + \gamma_{1,2}) - \gamma_{1,1}\gamma_{1,2} - \gamma_{2,1}\gamma_{2,2} + h(\gamma_{1,1} + \gamma_{1,2} - \gamma_{2,1} - \gamma_{2,2}) + 2h^2) + (z_3z_4 + z_2z_3 + z_2z_4 + z_1z_2 + z_1z_3 + z_1z_4)(\mathbf{q} - 1)$$

$$p_3 = -(\gamma_{1,1} + \gamma_{1,2})\gamma_{2,1}\gamma_{2,2} - \gamma_{1,1}\gamma_{1,2}(\gamma_{2,1} + \gamma_{2,2}) \quad (\text{A.4})$$

$$+ \mathbf{q}(\gamma_{1,1} + \gamma_{1,2})\gamma_{2,1}\gamma_{2,2} + \gamma_{1,1}\gamma_{1,2}(\gamma_{2,1} + \gamma_{2,2}) - h^2(\gamma_{1,1} + \gamma_{1,2} + \gamma_{2,1} + \gamma_{2,2}) - 2h(\gamma_{1,1}\gamma_{1,2} - \gamma_{2,1}\gamma_{2,2}) - (z_2z_3z_4 + z_1z_3z_4 + z_1z_2z_3 + z_1z_2z_4)(\mathbf{q} - 1)$$

$$p_4 = \gamma_{1,1}\gamma_{1,2}\gamma_{2,1}\gamma_{2,2} - \mathbf{q}(\gamma_{1,1} + h)(\gamma_{1,2} + h)(h - \gamma_{2,1})(h - \gamma_{2,2}) + z_1z_2z_3z_4(\mathbf{q} - 1) \quad (\text{A.5})$$

We solve two of above four equations $p_l = 0$, $l = 1, 2, 3, 4$ for $s_1 = \gamma_{2,1} + \gamma_{2,2}$ and $s_2 = \gamma_{2,1}\gamma_{2,2}$ and plug the solution into the rest two equations. Then four independent equations are obtained. From these four equations, with change of variables in terms of symmetric polynomials

$$Z_1 := z_1 + z_2 + z_3 + z_4, \quad Z_2 := z_1z_2 + z_3z_2 + z_4z_2 + z_1z_3 + z_1z_4 + z_3z_4, \quad (\text{A.6})$$

$$Z_3 := z_1z_2z_3 + z_1z_4z_3 + z_2z_4z_3 + z_1z_2z_4, \quad Z_4 := z_1z_2z_3z_4$$

$$X_1 := \gamma_{1,1} + \gamma_{1,2}, \quad X_2 := \gamma_{1,1}\gamma_{1,2} \quad (\text{A.7})$$

we obtain (3.36)

$$X_1^3 = -\frac{hq(4h^2q + Z_1(h - 3hq) + 2(q - 1)Z_2)}{(q - 1)^2} + Z_3 + X_1 \left(\frac{hq(h(2 - 6q) + 3(q - 1)Z_1)}{(q - 1)^2} - Z_2 \right) + X_2 \left(\frac{4hq}{q - 1} - Z_1 \right) + X_1^2 \left(Z_1 - \frac{4hq}{q - 1} \right) + 2X_1X_2 \quad (\text{A.8})$$

$$X_1^2X_2 = \frac{hq(h^3(q(q + 4) - 1) - h^2(q(q + 2) - 1)Z_1 + h(q^2 - 1)Z_2 - (q - 1)^2Z_3) + (q - 1)^3Z_4}{(q - 1)^3} + X_1 \frac{2h^2q(-3hq + h + (q - 1)Z_1)}{(1 - q)^3} + X_2 \left(\frac{hq(h(2 - 6q) + 3(q - 1)Z_1)}{(q - 1)^2} - Z_2 \right) + \frac{2h^2qX_1^2}{(q - 1)^2} + X_1X_2 \left(Z_1 - \frac{4hq}{q - 1} \right) + X_2^2 \quad (\text{A.9})$$

$$X_1X_2^2 = \frac{2h^2q((q - 1)^2Z_3 - hq(4h^2q + Z_1(h - 3hq) + 2(q - 1)Z_2))}{(q - 1)^4} \quad (\text{A.10})$$

$$\begin{aligned} & + X_1 \frac{(hq(h^3(q((q - 9)q - 1) + 1) - h^2(q((q - 5)q + 3) + 1)Z_1 + h(q - 1)^3Z_2 - (q - 1)^3Z_3) + (q - 1)^4Z_4)}{(q - 1)^4} \\ & + X_2 \frac{(hq(4h^2q(q + 1) + h(q(2 - 3q) + 1)Z_1 + 2(q - 1)^2Z_2) - (q - 1)^3Z_3)}{(q - 1)^3} - X_1^2 \frac{2h^3q(q + 1)}{(q - 1)^3} + X_1X_2 \frac{4h^2q}{(q - 1)^2} + X_2^2 \left(Z_1 - \frac{4hq}{q - 1} \right) \\ X_2^3 = & \frac{2h^2q(h(h(hq(h(q(5q + 8) - 1) + (2 - 4q(q + 1))Z_1) + 3q(q^2 - 1)Z_2) - (q - 1)^2(2q + 1)Z_3) + (q - 1)^3Z_4)}{(q - 1)^5} \quad (\text{A.11}) \\ & + X_1 \frac{2h^2q(h(2hq(h(q(q + 7) - 2) - 3(q - 1)Z_1) - (q - 1)^3Z_2) + (q - 1)^3Z_3)}{(q - 1)^5} \\ & + X_2 \frac{(hq(-h^3(q(q(q + 23) - 1) + 1) + h^2(q - 1)(q(q + 10) + 1)Z_1 - h(q - 1)^2(q + 3)Z_2 + (q - 1)^3Z_3) - (q - 1)^4Z_4)}{(q - 1)^4} \\ & + X_1^2 \frac{(h^4q(q((q - 1)q + 11) + 1) - h(q - 1)^3q(h^2Z_1 - hZ_2 + Z_3) + (q - 2)q((q - 2)q + 2)Z_4 + Z_4)}{(q - 1)^4} \\ & + X_1X_2 \frac{(hq(2h^2(q(2q - 7) - 1) - 3h(q - 1)^2Z_1 + 2(q - 1)^2Z_2) - (q - 1)^3Z_3)}{(q - 1)^3} + X_2^2 \left(\frac{3hq(2hq - (q - 1)Z_1)}{(q - 1)^2} + Z_2 \right) \end{aligned}$$

References

- [1] N. A. Nekrasov and S. L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, *Nucl. Phys. Proc. Suppl.* **192-193** (2009) 91–112, [[0901.4744](#)].
- [2] N. A. Nekrasov and S. L. Shatashvili, *Quantum integrability and supersymmetric vacua*, *Prog. Theor. Phys. Suppl.* **177** (2009) 105–119, [[0901.4748](#)].
- [3] C. Closset, S. Cremonesi and D. S. Park, *The equivariant A-twist and gauged linear sigma models on the two-sphere*, *JHEP* **06** (2015) 076, [[1504.06308](#)].
- [4] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, *JHEP* **07** (2015) 127, [[1504.03698](#)].
- [5] K. Ohta and Y. Yoshida, *Non-abelian localization for supersymmetric yang-mills-chern-simons theories on seifert manifold*, [1205.0046v3](#).
- [6] C. Closset and I. Shamir, *The $\mathcal{N} = 1$ Chiral Multiplet on $T^2 \times S^2$ and Supersymmetric Localization*, *JHEP* **03** (2014) 040, [[1311.2430](#)].
- [7] T. Nishioka and I. Yaakov, *Generalized indices for $\mathcal{N} = 1$ theories in four-dimensions*, *JHEP* **12** (2014) 150, [[1407.8520](#)].
- [8] M. Honda and Y. Yoshida, *Supersymmetric index on $T^2 \times S^2$ and elliptic genus*, [1504.04355](#).
- [9] V. Gorbounov, R. Rimányi, V. Tarasov and A. Varchenko, *Quantum cohomology of the cotangent bundle of a flag variety as a yangian bethe algebra*, *Journal of Geometry and Physics* **74** (2013) 56 – 86.
- [10] R. Rimányi, V. Tarasov and A. Varchenko, *Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety*, *Journal of Geometry and Physics* **94** (aug, 2015) 81–119, [[1411.0478](#)].
- [11] C. Closset and H. Kim, *To appear*, .
- [12] N. A. Nekrasov and S. L. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, in *Proceedings, 16th International Congress on Mathematical Physics (ICMP09)*, 2009. [0908.4052](#).
- [13] H. de Vega, *Families of commuting transfer matrices and integrable models with disorder*, *Nuclear Physics B* **240** (1984) 495 – 513.
- [14] V. E. Korepin, *Calculation of norms of bethe wave functions*, *Comm. Math. Phys.* **86** (1982) 391–418.
- [15] S. Okuda and Y. Yoshida, *G/G gauged WZW model and Bethe Ansatz for the phase model*, *JHEP* **11** (2012) 146, [[1209.3800](#)].
- [16] S. Okuda and Y. Yoshida, *G/G gauged WZW-matter model, Bethe Ansatz for q-boson model and Commutative Frobenius algebra*, *JHEP* **03** (2014) 003, [[1308.4608](#)].
- [17] S. Gukov and D. Pei, *Equivariant Verlinde formula from fivebranes and vortices*, [1501.01310](#).
- [18] S. Okuda and Y. Yoshida, *Gauge/Bethe correspondence on $S^1 \times \Sigma_h$ and index over moduli space*, [1501.03469](#).
- [19] N. A. Nekrasov and S. L. Shatashvili, *Bethe/Gauge correspondence on curved spaces*, *JHEP* **01** (2015) 100, [[1405.6046](#)].
- [20] D. R. Morrison and M. R. Plesser, *Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties*, *Nucl. Phys.* **B440** (1995) 279–354, [[hep-th/9412236](#)].

- [21] D. Maulik and A. Okounkov, *Quantum Groups and Quantum Cohomology*, [1211.1287](#).
- [22] E. Witten, *Phases of $N=2$ theories in two-dimensions*, *Nucl. Phys.* **B403** (1993) 159–222, [[hep-th/9301042](#)].
- [23] F. Benini, D. S. Park and P. Zhao, *Cluster Algebras from Dualities of $2d \mathcal{N} = (2, 2)$ Quiver Gauge Theories*, *Commun. Math. Phys.* **340** (2015) 47–104, [[1406.2699](#)].
- [24] C. Korff, *Cylindric versions of specialised macdonald functions and a deformed Verlinde algebra*, *Commun. Math. Phys.* **318** (2013) 173–246.
- [25] F. Benini and A. Zaffaroni, *Supersymmetric partition functions on Riemann surfaces*, [1605.06120](#).